THE ATIYAH-HITCHIN BRACKET FOR THE CUBIC NONLINEAR SCHRÓDINGER EQUATION. II. PERIODIC POTENTIALS.

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Abstract. This is the second in a series of papers on Poisson formalism for the cubic nonlinear Schrödinger equation with repulsive nonlinearity. In this paper we consider periodic potentials. The inverse spectral problem for the periodic auxiliary Dirac operator leads to a hyperelliptic Riemann surface Γ. Using the spectral problem we introduce on this Riemann surface a meromorphic function $\mathcal{X}$. We call it the Weyl function, since it is closely related to the classical Weyl function discussed in the first paper. We show that the pair $(\Gamma, \mathcal{X})$ carries a natural Poisson structure. We call it the deformed Atiyah–Hitchin bracket. The Poisson bracket on the phase space is the image of the deformed Atiyah–Hitchin bracket under the inverse spectral transform.

1. Introduction.

1.1. **Statement of the problem.** The fact that equations integrable by the method of the inverse spectral transform are Hamiltonian systems was realized in the very early days of the theory. Gardner, Zakharov and Faddeev [8, 29], found that the Korteveg de Vries equation [1]

$$u^\bullet = 6u'u - u''', \qquad\qquad u = u(x,t);$$

on the line with rapidly decaying initial data can be written as a Hamiltonian system

$$u^\bullet = \{u, \mathcal{H}\},$$

with Hamiltonian $\mathcal{H} = \int u^3 + \frac{1}{2}(u')^2\, dx$ and the bracket

$$\{A, B\} = \int \frac{\delta A}{\delta u(x)} \frac{\partial}{\partial x} \frac{\delta B}{\delta u(x)}\, dx. \tag{1.1}$$

Soon after Zakharov and Manakov, [28], integrated the nonlinear Schrödinger equation with repulsive nonlinearity

$$i\psi^\bullet = -\psi'' + 2|\psi|^2\psi,$$

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[1] Prime $'$ signifies the derivative in the variable x and dot $\bullet$ the derivative with respect to time.

where $\psi(x,t)$ is a smooth complex function on the line with rapid decay at infinity. The equation is a Hamiltonian system

$$\psi^{\bullet} = \{\psi, \mathcal{H}\},$$

with Hamiltonian $\mathcal{H} = \frac{1}{2}\int |\psi'|^2 + |\psi|^4 \, dx =$*energy* and the classical bracket

$$\{A, B\} = 2i \int \frac{\delta A}{\delta\overline{\psi}(x)} \frac{\delta B}{\delta\psi(x)} - \frac{\delta A}{\delta\psi(x)} \frac{\delta B}{\delta\overline{\psi}(x)} \, dx. \tag{1.2}$$

The NLS equation will surve as our model example. The KdV case is more subtle and will be considered in [26].

It was demonstrated by Novikov, Dubrovin, Its–Matveev and McKean–Moerbecke, [7, 18], that the periodic problem for the KdV equation is connected with hyperelliptic Riemann surfaces. The periodic problem for the cubic NLS is also connected with hyperelliptic Riemann surfaces, [10].

At the present time we know numerous examples of integrable systems and various Hamiltonian formulations of them. Until now it was not known how to obtaining the *Poisson formalism* from the corresponding Riemann surface. The goal of the present paper is to make a new step in this direction. Namely we relate bracket 1.2 with the Poisson structure on the meromorphic functions defined on the corresponding hyperelliptic Riemann surfaces. This Poisson bracket we call the deformed Atiyah–Hitchin bracket.

In our previous paper [22] we introduced the spectral cover for a class of general potentials. This is two sheeted covering of the complex plane cut along the real line. The spectral cover is an open manifold which consist of four (disconnected) copies of the complex half–plane. On this cover we defined the classical Weyl function. We showed that the Weyl function carries the Atiyah–Hitchin bracket. The formula for the bracket has a different shape for different parts of the cover.

The construction of the present paper is a compactification of the spectral cover. In the case of periodic finite gap potentials the spectral cover can be glued into a plane curve biholomorphicaly equivalent to a compact smooth hyperelliptic Riemann surface. The Weyl function can be analytically extended across the glued edges. Now the spectral cover is simply connected and the Poisson bracket on Weyl functions is described by a single formula. This is the deformed Atiyah–Hitchin bracket. We demonstrate that the Poisson bracket on the phase space is the image of this deformed AH bracket under the inverse spectral transform.

We would like to make one historical remark. Novikov and Veselov in their pioneering paper [20] singled out a class of brackets for the KdV equation. They call these brackets *analytic brackets compatible with algebraic geometry.* The brackets can be written in terms of singularities of the Floquet solutions or, in other words, eigenfunctions of the auxiliary spectral problem with special monodromy properties. The Gardner bracket 1.1, the Lenard-Magri bracket [15], *etc.*, are examples of such brackets. Similar situation was noted for the NLS hierarchy, [6].

The approach of this paper is conceptually different from the approach of Novikov *at all*. We consider the bracket where the corresponding functions are holomorphic, while Novikov *at all* express the brackets in terms of singularities. Nevertheless the term *analytic brackets compatible with algebraic geometry* coined by them describes what happens here in the most precise way.

In the rest of the introduction we describe our strategy and results. We try to sweep all technicalities under the rug in order to give a reader a clear geometrical picture.

1.2. **The Atiyah-Hitchin bracket.** Atiyah and Hitchin, [1], introduced a symplectic structure on the space of meromorphic maps $\mathcal{X}(\lambda) : \mathbb{CP}^1 \to \mathbb{CP}^1$ of the form

$$\mathcal{X}(\lambda) = -\frac{q(\lambda)}{p(\lambda)},$$

where $q(\lambda)$ is a polynomial of degree $N-1$ and $p(\lambda)$ is a monic polynomial of degree N with distinct roots. The parameters $\lambda_1, \dots, \lambda_N, q(\lambda_1), \dots, q(\lambda_N)$, are complex coordinates on this space and δ denotes a variation of these coordinates. The Atiyah–Hitchin nondegenerate close 2-form ω is defined by the formula

$$\omega = \sum_{k=1}^{N} \frac{\delta q(\lambda_k)}{q(\lambda_k)} \wedge \delta\lambda_k.$$

The corresponding Poisson bracket is specified by canonical relations:

$$\{q(\lambda_n), \lambda_k\} = \delta_k^n \, q(\lambda_n); \qquad \{\lambda_n, \lambda_k\} = \{q(\lambda_n), q(\lambda_k)\} = 0.$$

The bracket turns the space of maps $\mathcal{X}(\lambda) : \mathbb{CP}^1 \to \mathbb{CP}^1$ into a Poisson manifold. Consider $\mathcal{X}(\lambda)$ and $\mathcal{X}(\mu)$ where the variables λ and μ are fixed and away from the poles. Then $\mathcal{X}(\lambda)$ and $\mathcal{X}(\mu)$ (considered as functions of $\mathcal{X}$) are functions of coordinates $\lambda_1, \dots, \lambda_N$, $q(\lambda_1), \dots, q(\lambda_N)$. As it was demonstrated by Faybusovich and Gekhtman, [2], the bracket for $\mathcal{X}(\lambda)$ and $\mathcal{X}(\mu)$, is given by the formula

$$\{\mathcal{X}(\lambda), \mathcal{X}(\mu)\} = \frac{(\mathcal{X}(\lambda) - \mathcal{X}(\mu))^2}{\lambda - \mu}. \tag{1.3}$$

This formula is much more general than its coordinate version, see [22].

In this paper we construct Poisson structure on the space of pairs $(\Gamma, \mathcal{X})$ where Γ is a hyperelliptic Riemann surface of infinite genus associated with an inverse spectral problem for the Dirac operator

$$\mathfrak{D}\boldsymbol{f} = \left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} i\partial_x + \begin{pmatrix} 0 & -i\overline{\psi} \\ i\psi & 0 \end{pmatrix} \right] \boldsymbol{f} = \frac{\lambda}{2}\boldsymbol{f}, \qquad \boldsymbol{f} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \tag{1.4}$$

in the class of all smooth periodic potentials $\psi(x + 2l) = \psi(x)$ and $\mathcal{X} : \Gamma \to \mathbb{CP}^1$ is the Weyl function. An isospectral deformations of the Dirac operator (which

preserve Γ) are the flows of cubic nonlinear Schrödinger hierarchy. We construct the Poisson structure on $(\Gamma, \mathcal{X})$ using the Poisson bracket 1.2.

We illustrate our strategy using the Camassa–Holm equation and the open Toda lattice. In both cases the Riemann surface is reducible with components being copies of $\mathbb{CP}^1$. Each component of such curve has a global uniformization parameter. These curves with nodal singularities appear in the compactification[2] of the space of smooth hyperelliptic curves (possibly of infinite genus). This is a great simplification of the hyperelliptic case since each component can be treated by analysis methods. Such reducible curves were first considered in the theory of completely integrable systems by McKean in the beginning of 1980's, [16]. The Baker-Akhiezer function for such curves was introduced only recently by Krichever and the author, [13]. We describe the Poisson brackets for the Weyl functions defined on these boundary curves.

Example 1.1. The Camassa–Holm equation.

The simplest situation when formula 1.3 appears is the Camassa–Holm equation

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + \frac{\partial}{\partial x} G\left[v^2 + \frac{1}{2}\left(\frac{\partial v}{\partial x}\right)^2\right] = 0$$

in which $t \geq 0$ and $-\infty < x < \infty$, $v = v(x,t)$ is velocity, and G is inverse to $1 - d^2/dx^2$ *i.e.*,

$$G : f(x) \to \frac{1}{2}\int\limits_{-\infty}^{+\infty} e^{-|x-y|} f(y)dy.$$

The CH equation can be formulated as a Hamiltonian system

$$m^\bullet + \{m, \mathcal{H}\} = 0, \qquad\qquad m = v - \frac{\partial^2 v}{\partial x^2};$$

with the Hamiltonian

$$\mathcal{H} = \frac{1}{2}\int_{-\infty}^{+\infty} mv\, dx = \text{energy}$$

and the bracket

$$\{A, B\} = \int_{-\infty}^{+\infty} \frac{\delta A}{\delta m}\,(mD + Dm)\,\frac{\delta B}{\delta m}\, dx. \tag{1.5}$$

The CH equation preserves the Dirichlet spectrum of the string spectral problem,

$$\frac{\partial^2 f(\xi)}{\partial \xi^2} + \lambda g(\xi) f(\xi) = 0, \qquad\qquad -2 \leq \xi \leq 2.$$

[2]The situation is similar to the Deligne-Mumford compactification of the space of smooth curves by stable curves, [5].

The variables ξ and x are related by

$$x \longrightarrow \xi = 2\tanh\frac{x}{2}.$$

Also the potential $g(\xi)$ is related to $m(x)$ by the formula $g(\xi) = m(x)\cosh^4\frac{x}{2}$.

The Riemann surface Γ associated with the spectral problem consists of two components Γ_- and Γ_+ (see Figure 1), two copies of $\mathbb{CP}^1$. The components are glued to each other at the points of the Dirichlet spectrum $\lambda_k,\ k = 1, 2, \ldots$. A point on the curve is denoted by $Q = (\lambda, \pm)$, where λ is the spectral parameter and the sign $\pm$ refers to the component. An infinities of $\Gamma_\pm$ are denoted by $P_\pm$ correspondingly.

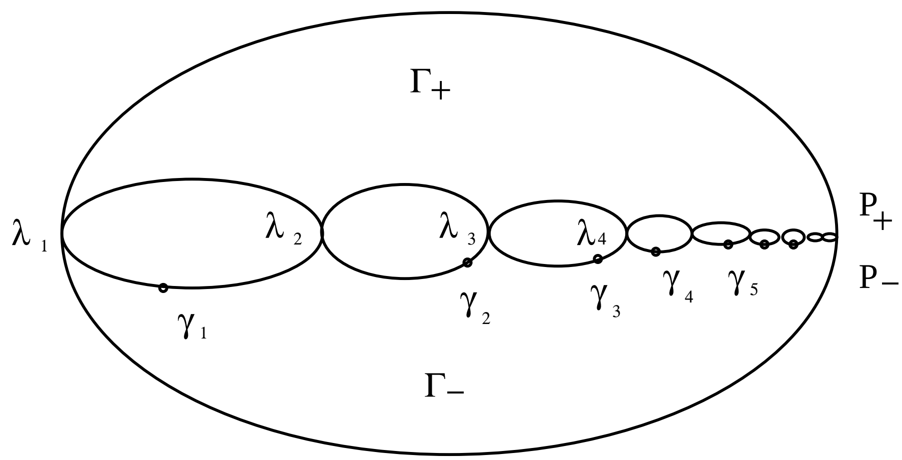

FIGURE 1. The reducible Riemann surface with infinitely many nodal points for the CH equation.

The Baker–Akhiezer function $e(\xi, Q)$ is a function on Γ which depends on the variable $\xi, -2 \le \xi \le +2$, as a parameter. In this parameter the BA function is a solution of the string spectral problem with $\lambda = \lambda(Q)$. The BA function is holomorphic outside simple poles at the points of the divisor $\boldsymbol{\gamma}_k = (\mu_k, -),\ k = 1, 2, \ldots$, and singularities at infinities $P_\pm$. The Baker–Akhiezer function satisfies the gluing condition

$$e(\xi, (\lambda_k, +)) = e(\xi, (\lambda_k, -)), \qquad\qquad k = 1, 2, \ldots.$$

For $Q \in \Gamma_-$ we define the Weyl function by the formula

$$\mathcal{X}(\lambda) = \frac{\partial}{\partial\xi}\log e(\xi, Q)|_{\xi=-2}, \qquad\qquad \lambda = \lambda(Q).$$

The function $\mathcal{X}$ is a real meromorphic map $\mathbb{CP}^1 \to \mathbb{CP}^1$. After a change of the spectral parameter $\lambda \to -1/\lambda$ the function $\mathcal{X}$ takes the form

$$\mathcal{X}(\lambda) = -\frac{1}{4} + \sum_{k=1}^{\infty} \frac{\rho_k}{\lambda_k - \lambda}.$$

The Poisson bracket 1.5 computed for $\mathcal{X}$ is given by formula 1.3. We refer to [24] for details.

Example 1.2. The open Toda hierarchy.

This N–particles system is another example where the space of maps $\mathbb{CP}^1 \to \mathbb{CP}^1$ occurs. The Hamiltonian of the system is

$$H = \sum_{k=0}^{N-1} \frac{p_k^2}{2} + \sum_{k=0}^{N-2} e^{q_k - q_{k+1}}.$$

Introducing the classical Poisson bracket

$$\{f, g\} = \sum_{k=0}^{N-1} \frac{\partial f}{\partial q_k}\frac{\partial g}{\partial p_k} - \frac{\partial f}{\partial p_k}\frac{\partial g}{\partial q_k}, \tag{1.6}$$

we write the equations of motion as

$$\begin{aligned} \overset{\bullet}{q_k} &= \{q_k, H\} = p_k, \\ \overset{\bullet}{p_k} &= \{p_k, H\} = -e^{q_k - q_{k+1}} + e^{q_{k-1} - q_k}, \qquad k = 0, \dots, N-1. \end{aligned}$$

We put $q_{-1} = -\infty,\ q_N = \infty$ in all formulas.

The Toda flow preserves the spectrum $\lambda_1 < \dots < \lambda_N$ of the three diagonal Jacobi matrix

$$L = \begin{bmatrix} v_0 & c_0 & 0 & \cdots & 0 \\ c_0 & v_1 & c_1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdots & c_{N-3} & v_{N-2} & c_{N-2} \\ 0 & \cdots & 0 & c_{N-2} & v_{N-1} \end{bmatrix},$$

where

$$c_k = e^{q_k - q_{k+1}/2}, \qquad v_k = -p_k.$$

The Riemann surface Γ associated with this spectral problem consists of two components Γ_- and Γ_+, two copies of $\mathbb{CP}^1$. The components are glued to each other at the points of the spectrum $\lambda_k,\ k = 1, \dots, N$; (see Figure 2).

The Baker–Akhiezer function $e(n, Q)$ is a function on Γ which depends on the variable n, $n = 1, \dots, N$, as a parameter. In this parameter it satisfies the three term reccurence relation produced by the matrix L. The BA function is holomorphic outside simple poles at the points of the divisor $\boldsymbol{\gamma}_k = (\mu_k, -),\ k = 1, \dots, N-1;$

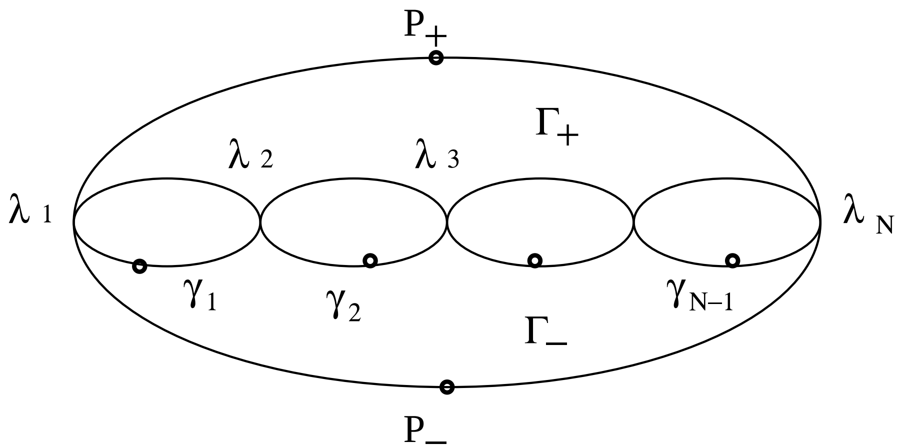

FIGURE 2. The reducible Riemann surface with finite number of nodal points for the open Toda.

and multiple poles at infinities $P_\pm$. The Baker–Akhiezer function satisfies the gluing condition

$$e(n,(\lambda_k,+)) = e(n,(\lambda_k,-)), \qquad k = 1,\dots,N.$$

When $Q \in \Gamma_-$ we define the Weyl function by the formula

$$\mathcal{X}(\lambda) = \frac{1}{c_0 e(1,Q) - \lambda + v_0}, \qquad \lambda = \lambda(Q).$$

The function $\mathcal{X}$ is a rational map $\mathbb{CP}^1 \to \mathbb{CP}^1$. It can be expanded in simple fractions

$$\mathcal{X}(\lambda) = \frac{\rho_1}{\lambda_1 - \lambda} + \dots + \frac{\rho_N}{\lambda_N - \lambda},$$

with λ_k being real and $\rho_k > 0$. Moreover, the spectral problem imposes an additional condition $\sum \rho_k = 1$ or equivalently

$$\operatorname{res}_{P_-} \mathcal{X}(\lambda) d\lambda = 1. \tag{1.7}$$

The Poisson bracket 1.6 computed for $\mathcal{X}$ takes the form

$$\{\mathcal{X}(\lambda), \mathcal{X}(\mu)\} = (\mathcal{X}(\lambda) - \mathcal{X}(\mu)) \left(\frac{\mathcal{X}(\lambda) - \mathcal{X}(\mu)}{\lambda - \mu} - \mathcal{X}(\lambda)\mathcal{X}(\mu) \right). \tag{1.8}$$

This bracket is a Dirac restriction of the AH bracket to the submanifold 1.7. The bracket 1.8 is degenerate with the Casimir

$$\sum \lambda_k = -\sum p_k.$$

For the unrestricted AH bracket this quantity is canonically conjugate to $\sum \rho_k$, see [23] for details.

1.3. **The deformed Atiyah-Hitchin bracket.** These examples illustrate the following general scheme. We associate to every point of the phase space $\mathcal{M}$ of an integrable system a set of algebraic-geometrical data. These data are the Riemann surface Γ and the meromorphic function $\mathcal{X} : \Gamma \to \mathbb{CP}^1$,

$$\mathcal{M} \longrightarrow (\Gamma, \mathcal{X}).$$

We call this map the direct spectral transform.

For a generic periodic potential of the Dirac spectral problem 1.4 one has to consider curves of infinite genus. For the so–called finite gap potentials we describe the finite genus curves in the image of the direct spectral transform in purely geometrical terms. Generic infinite gap case has little to add to this picture.

The nonsingular curve [3] Γ of genus g has a meromorphic function $\lambda(Q)$, $Q \in \Gamma$ of degree two with two simple poles at two points P_- and P_+ which we call infinities. Thus the curve is hyperelliptic. Namely, there are $2g+2$ critical points $\boldsymbol{\lambda}_k^{\pm}$, $k = 1, \dots, g+1$, of the function $\lambda(Q)$. The critical values $\lambda_k^{\pm} = \lambda(\boldsymbol{\lambda}_k^{\pm})$ are points of the simple periodic /antiperiodic spectrum of the Dirac operator and branch points of this surface. The curve is defined by

$$\Gamma = \{(\lambda, R) \in \mathbb{C}^2 : \ R^2 = -\prod_{k=1}^{2g+2} (\lambda - \lambda_k)\}.$$

The self-adjoint Dirac spectral problem requires the branch points of Γ to be real. The hyperelliptic Riemann surface Γ has a standard anti–holomorphic involution $\epsilon_a : \Gamma \to \Gamma$ permuting infinities. The branch points are fixed points of this involution :

$$\epsilon_a \boldsymbol{\lambda}_k^{\pm} = \boldsymbol{\lambda}_k^{\pm}.$$

The branch points lie on the fixed real ovals of the anti–holomorphic involution. There are $g+1$ real ovals $a_1, \dots, a_{g+1}$ (see Figure 3).

The spectral problem imposes another restriction on the corresponding hyperellitic Riemann surfaces. The Riemann surface caries a Floquet muliplier $w(Q)$. The transcendental function $w(Q)$ is holomorphic in the finite part of the curve and $w(Q) \sim e^{\pm il\lambda(Q)}$ for Q near $P_{\pm}$. At the branch points $w(\boldsymbol{\lambda}_k^{\pm}) = \pm 1$. The function $w(Q)$ takes values ± 1 also at the points of the double spectrum.

Now when we are back to the generic potentials we describe the function $\mathcal{X}$ which we call the Weyl function on the Riemann surface. The spectral problem 1.4 determines the 2×2 monodromy matrix for the shift $x \longrightarrow x + 2l$ over the period of the potential. The eigenvector of the monodromy matrix trivializes on the spectral curve Γ. The eigenvector is defined up to a multiplicative constant, but the ratio of its components is defined uniquely. This ratio is the meromorphic function $\mathcal{X}(x, Q)$, $Q \in \Gamma$, and can be expressed in terms of the Baker-Akhiezer

[3] The curve we are considering here is a resolution of singularities for some plane curve with infinitely many intersection points.

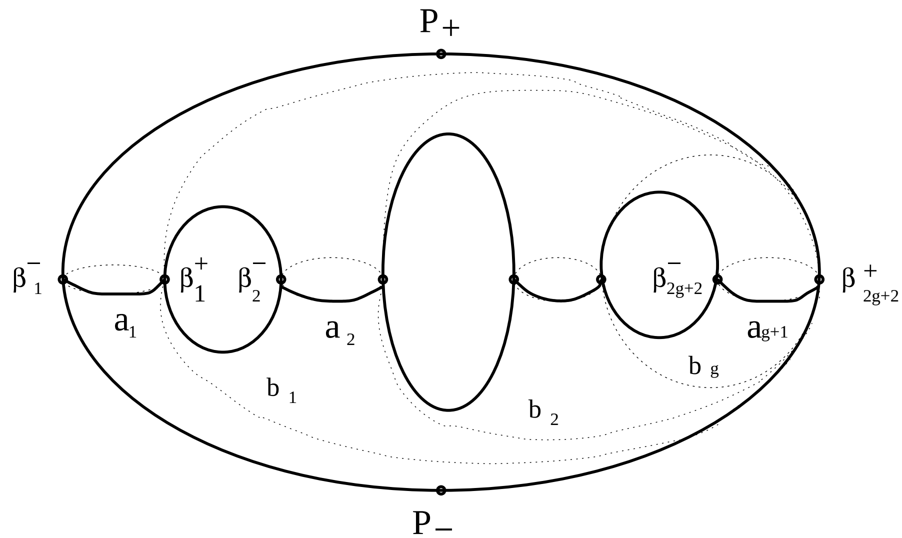

FIGURE 3. The hyperelliptic Riemann surface.

function. The ratio plays the role of the logarithmic derivative that we used for the CH and Toda lattice.

The direct spectral transform maps the Poisson bracket 1.2 defined on the phase space $\mathcal{M}$ of all smooth $2l$-periodic potentials to the space of pairs $(\Gamma, \mathcal{X})$. This bracket on the target space $(\Gamma, \mathcal{X})$ we call the deformed AH bracket. To state the formula for it we introduce the deformation factor

$$\Omega(Q) = \frac{w^2(Q) + 1}{w^2(Q) - 1}.$$

At the branch or intersection points the function Ω has poles because $w^2(Q) = 1$. The deformed AH bracket for two functions $\mathcal{X}(Q) = \mathcal{X}(x, Q)$ and $\mathcal{X}(P) = \mathcal{X}(x, P)$ is given by the formula

$$\{\mathcal{X}(Q), \mathcal{X}(P)\} = -2 \times \frac{(\mathcal{X}(Q) - \mathcal{X}(P))^2}{\lambda - \mu} \times \frac{\Omega(Q) + \Omega(P)}{2}.$$

Or course, we assume that Q and P are not at the poles of the function Ω or $\mathcal{X}$.

The formula differs from the "pure" AH bracket 1.3 by multiplication on a sum of deformation factors at two points P and Q. The origin of the deformation factor lies in the fact that Γ covers the plane of the spectral parameter $2:1$. At the branch or intersection points the function $\lambda(Q)$ can not be taken as a local parameter and the deformation factor remedies the situation.

The direct spectral transform can be inverted. The inverse map

$$(\Gamma, \mathcal{X}) \longrightarrow \mathcal{M}$$

is called the inverse spectral transform. The image of the deformed AH on the phase space under the inverse spectral transform is the standard Poisson bracket 1.2. Therefore, the deformed AH bracket can be taken as a starting formula for construction of the Poisson formalism[4].

1.4. **Content of the paper.** The paper is divided into two parts. In Section 2 we discuss the properties of the direct and inverse spectral transform. In Section 3 we compute the image of the bracket for the direct and inverse spectral transform. We also discuss the construction of the canonical variables.

Finally, the author would like to thank I. Krichever, S. Novikov, S. Natanzon, H. McKean and M. Shapiro for stimulating discussions.

2. The Spectral Problem.

2.1. **The NLS hierarchy.** The NLS equation

$$i\psi^{\bullet} = -\psi'' + 2|\psi|^2\psi, \tag{2.1}$$

where $\psi(x,t)$ is a smooth $2l$–periodic, $\psi(x+2l) = \psi(x)$, complex function is a Hamiltonian system

$$\psi^{\bullet} = \{\psi, \mathcal{H}\},$$

with Hamiltonian $\mathcal{H} = \frac{1}{2}\int_{-l}^{l} |\psi'|^2 + |\psi|^4\, dx =$*energy* and the bracket

$$\{A, B\} = 2i \int \frac{\delta A}{\delta\overline{\psi}(x)}\frac{\delta B}{\delta\psi(x)} - \frac{\delta A}{\delta\psi(x)}\frac{\delta B}{\delta\overline{\psi}(x)}\, dx. \tag{2.2}$$

The NLS Hamiltonian $\mathcal{H} = \mathcal{H}_3$ is one in the infinite series of commuting integrals of motion

$$\begin{aligned}
\mathcal{H}_1 &= \frac{1}{2}\int_{-l}^{l} |\psi|^2 dx,\\
\mathcal{H}_2 &= \frac{1}{2i}\int_{-l}^{l} \psi\bar{\psi}' dx,\\
\mathcal{H}_3 &= \frac{1}{2}\int_{-l}^{l} |\psi'|^2 + |\psi|^4\, dx, \quad \textit{etc.}
\end{aligned}$$

Hamiltonians produce an infinite hierarchy of flows $e^{tX_m},\ m = 1, 2, \ldots$.

The NLS equation 2.1 is a compatibility condition for the commutator

$$[\partial_t - V_3, \partial_x - V_2] = 0, \tag{2.3}$$

[4]The same formula produces the Gardner–Zakharov–Faddeev bracket 1.1 for the KdV equation, [26].

with[5]

$$V_2 = -\frac{i\lambda}{2}\sigma_3 + Y_0 = \begin{pmatrix} -\frac{i\lambda}{2} & 0 \\ 0 & \frac{i\lambda}{2} \end{pmatrix} + \begin{pmatrix} 0 & \overline{\psi} \\ \psi & 0 \end{pmatrix}$$

and

$$V_3 = \frac{\lambda^2}{2} i\sigma_3 - \lambda Y_0 + |\psi|^2 i\sigma_3 - i\sigma_3 Y_0'.$$

We often omit the lower index and write $V = V_2$. Each flow e^{tX_m} of the hierarchy can be written in the form 2.3 with the suitable operator $\partial_t - V_m$.

2.2. **The direct spectral problem for the Dirac operator.** We cite here freely the results of [17]. We assume that the periodic potential $\psi(x)$ is defined on the entire line. The commutator relation produces the auxiliary linear problem

$$\boldsymbol{f}'(x,\lambda) = V\boldsymbol{f}(x,\lambda), \qquad \boldsymbol{f}(x,\lambda) = \begin{bmatrix} f_1(x,\lambda) \\ f_2(x,\lambda) \end{bmatrix}. \tag{2.4}$$

This can be written as an eigenvalue problem for the Dirac operator

$$\mathfrak{D}\boldsymbol{f} = \left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} i\partial_x + \begin{pmatrix} 0 & -i\overline{\psi} \\ i\psi & 0 \end{pmatrix} \right] \boldsymbol{f} = \frac{\lambda}{2}\boldsymbol{f}. \tag{2.5}$$

We introduce a 2×2 transition matrix $M(x, y, \lambda)$:

$$M(x,y,\lambda) = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} (x,y,\lambda),$$

which is a solution of the equation

$$M'(x,y,\lambda) = VM(x,y,\lambda), \qquad M(y,y,\lambda) = I.$$

Let us define the monodromy matrix

$$M(x,\lambda) = M(x+2l,x,\lambda) = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} (x,\lambda).$$

We have obvious relation

$$M(x,\lambda) = M(x+2l,l,\lambda)M(-l,\lambda)M^{-1}(x+2l,l,\lambda).$$

Therefore, all matrices $M(x,\lambda)$ are similar to $M(-l,\lambda)$. The monodromy matrix $M(-l,\lambda)$ is unimodular because V is traceless. The quantity $\Delta(\lambda) = \frac{1}{2}\mathrm{trace} M(-l,\lambda)$ is called a discriminant. The symmetry of the matrix $V(x,\lambda)$

$$\sigma_1 \overline{V}(x,\lambda)\sigma_1 = V(x,\bar{\lambda})$$

[5]Here and below σ denotes the *Pauli matrices*

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

produces the same relation for the transition matrix

$$\sigma_1 \overline{M}(x, y, \lambda)\sigma_1 = M(x, y, \bar{\lambda}). \tag{2.6}$$

This implies in particular $\overline{\Delta}(\lambda) = \Delta(\bar{\lambda})$ and $\Delta(\lambda)$ is real for real λ.

The eigenvalues of the monodromy matrix are called the Floquet multipliers. They are the roots of the quadratic equation

$$w^2 - 2\Delta w + 1 = 0, \tag{2.7}$$

and given by the formula

$$w = \Delta + \sqrt{\Delta^2 - 1}. \tag{2.8}$$

The values of $\lambda : w(\lambda) = \pm 1$ constitute the points of the periodic/antiperiodic spectrum. This condition is equivalent to $\Delta(\lambda) = \pm 1$. The self–adjointness of the Dirac operator 2.5 implies that points of the spectra are real. The NLS hierarchy preserves the periodic/antiperiodic spectrum.

Example 2.1. The monodromy matrix for the trivial potential $\psi \equiv 0$.

Let $\psi \equiv 0$. The corresponding transition matrix can be easily computed $M(x, y, \lambda) = e^{-i\frac{\lambda}{2}\sigma_3(x-y)}$. We have $\Delta(\lambda) = \cos \lambda l$ and double eigenvalues at the points $\lambda_n^{\pm} = \frac{\pi n}{l}$. If n is even/odd, then the corresponding $\lambda_n^{\pm}$ belongs to the periodic/ anti-periodic spectrum.

The Floquet multipliers become single–valued on the spectral curve

$$\Gamma = \{Q = (\lambda, w) \in \mathbb{C}^2 : \quad R(\lambda, w) = \det\left[M(-l, \lambda) - wI\right] = 0\}.$$

The plane curve consists of a two sheets covering the plane of the spectral parameter λ. For a generic periodic potential from the phase space $\mathcal{M}$ the genus of this curve if infinite, [17]. To describe geometry of Γ we consider the class of finite gap potentials. We assume that there are a finite number, namely $g + 1$, open gaps in the spectrum

$$\ldots < \lambda_{n-1}^- = \lambda_{n-1}^+ < \lambda_n^- < \lambda_n^+ < \ldots < \lambda_{n+g}^- < \lambda_{n+g}^+ < \lambda_{n+g+1}^- = \lambda_{n+g+1}^+ < \ldots$$

Figure 4 below presents an exampe of the discriminant $\Delta(\lambda)$ for a 3 gap potential.

Example 2.2. The Riemann surface for the trivial potential $\psi \equiv 0$.

We have $\Delta(\lambda) = \cos \lambda l$ and the quadratic equation 2.7 has the solutions $w(\lambda) = e^{\pm il\lambda}$. The Riemann surface $\Gamma = \Gamma_+ + \Gamma_-$ is reducible and consists of two copies of $\mathbb{CP}^1$ which intersect each other at the points of the double spectrum $\lambda_n^{\pm}$. Each part Γ_+ or Γ_- has the corresponding infinity P_+ or P_-. The Floquet multipliers are single valued on Γ:

$$\begin{aligned} w(Q) &= e^{+i\lambda l}, \qquad && Q \in \Gamma_+; \\ w(Q) &= e^{-i\lambda l}, \qquad && Q \in \Gamma_-. \end{aligned}$$

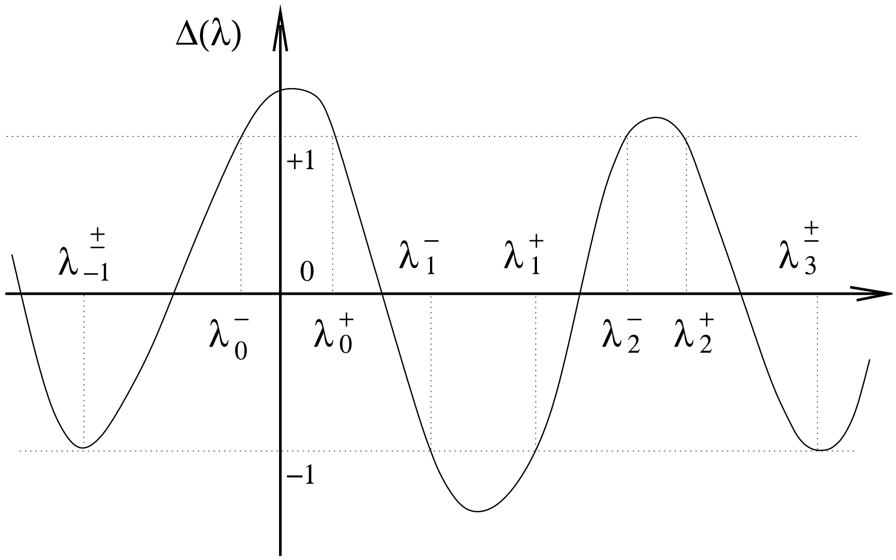

FIGURE 4. The discriminant $\Delta(\lambda)$ for 3 gap potential .

For a finite–gap potential the Riemann surface Γ is irreducible. The spectral curve is biholomorphicaly equivalent to the hyperelliptic curve with branch points at the points of the simple spectrum

$$\hat{\Gamma} = \{\hat{Q} = (\lambda, y) \in \mathbb{C}^2 : \quad \hat{R}(\lambda, y) = y^2 + \prod_{k=n}^{n+g} (\lambda_k^- - \lambda)(\lambda_k^+ - \lambda) = 0\}.$$

To establish the correspondence we need Hadamard product, see [17]:

$$\Delta^2(\lambda) - 1 = -\prod_{\mathbb{Z}} \frac{(\lambda_k^+ - \lambda)(\lambda_k^- - \lambda)}{a_k^2},$$

where $a_0 = \frac{1}{l}$ and $a_k = \frac{\pi k}{l}$, $k \neq 0$. The 1 to 1 map

$$\hat{Q} = (\lambda, y) \longrightarrow Q = (\lambda, w)$$

is defined by the formula

$$w = \Delta(\lambda) + \prod_{\mathbb{Z} - \{n, \ldots, n+g\}} \frac{(\lambda_k^{\pm} - \lambda)}{a_k} \times \prod_{\{n, \ldots, n+g\}} \frac{1}{a_k} \times y,$$

which follows from 2.8 and the Hadamard product. The Riemann surface $\hat{\Gamma}$ is the desingularization of the spectral curve Γ.

There are three possible types of important points on $\hat{\Gamma}$. These are the singular points, the points above $\lambda = \infty$ and the branch points which we now discuss in detail.

- The singular points are determined by the condition

$$\partial_\lambda \hat{R}(\lambda, y) = \partial_y \hat{R}(\lambda, y) = 0.$$

There are no singular points on $\hat{\Gamma}$.
- There are two nonsingular points $\hat{P}_+$ and $\hat{P}_-$ above $\lambda = \infty$. At these points[6]

$$w(\hat{Q}) = e^{+i\lambda l}\left(1 + O\left(1/\lambda\right)\right), \qquad \hat{Q} \in (\hat{P}_+); \tag{2.9}$$

$$w(\hat{Q}) = e^{-i\lambda l}\left(1 + O\left(1/\lambda\right)\right), \qquad \hat{Q} \in (\hat{P}_-). \tag{2.10}$$

- The branch points are specified by the condition

$$\partial_y \hat{R}(\lambda, y) = 0.$$

They are different from the singular points and correspond to the simple periodic/antiperiodic spectrum. We denote these points by $\hat{\boldsymbol{\lambda}}_k^{\pm} = (\lambda_k^{\pm}, 0)$, $k = n, \ldots, n+g$. There are $2(g+1)$ of them, each of which has a ramification index 1.

The Riemann-Hurwitz formula implies that the genus of the curve $\hat{\Gamma}$ is equal to g, one less than the number of open gaps in the spectrum.

The map between $\hat{\Gamma}$ and Γ transforms the infinities $\hat{P}_-$ and $\hat{P}_+$ into corresponding punctures P_- and P_+. It also maps the points $\hat{\boldsymbol{\lambda}}_k^{\pm}$ to $\boldsymbol{\lambda}_k^{\pm} = (\lambda_k^{\pm}, (-1)^k)$.

Remark 2.3. Contrary to the case of hyperelliptic surface $\hat{\Gamma}$, there are singular points on the plane curve Γ. These are the points $(\lambda_k^{\pm}, (-1)^k)$ of the double spectrum. At these points two sheets of the curve Γ intersect. These points accumulate at infinities $P_\pm$. Asymptotically they form an arithmetic sequence and approach infinities from *real* directions.

Let $\epsilon_\pm$ be a holomorphic involution on the curve Γ permuting sheets

$$\epsilon_\pm : \ (\lambda, w) \longrightarrow (\lambda, 1/w).$$

The fixed points of $\epsilon_\pm$ are the branch points of Γ. The involution $\epsilon_\pm$ permutes infinities $\epsilon_\pm : P_- \longrightarrow P_+$. Let us also define on Γ an antiholomorphic involution

$$\epsilon_a : (\lambda, w) \longrightarrow (\bar{\lambda}, \bar{w}).$$

The involution ϵ_a also permutes infinities and commutes with $\epsilon_\pm$. Points of the curve above open gaps $\left[\lambda_k^-, \lambda_k^+\right]$ form fixed "real" ovals of ϵ_a. There are $g+1$ real ovals $a_1, \ldots, a_{g+1}$.

Since $w(Q)$ never vanishes on Γ we can define the quasimomentum $p(Q)$ by the formula $w(Q) = e^{ip(Q)2l}$. Evidently, $p(Q)$ is defined up to $\frac{\pi n}{l}$, where n is an integer. The asymptotic expansion for $p(Q)$ at infinities can be easily computed

$$p(Q) = \pm\frac{\lambda}{2} + p_0^{\pm} \mp \frac{p_1}{\lambda} \mp \frac{p_2}{\lambda^2} \ldots, \qquad \lambda = \lambda(Q), \quad Q \in (P_\pm),$$

[6]The notation $\hat{Q} \in (\hat{P})$ means that the point $\hat{Q}$ is in the vicinity of the point $\hat{P}$.

where

$$p_0^{\pm} = \frac{\pi k_{\pm}}{l}, \qquad\qquad k_{\pm} \quad \text{is an integer}, \tag{2.11}$$

and

$$p_1 = \frac{1}{l}\mathcal{H}_1, \quad p_2 = \frac{1}{l}\mathcal{H}_2, \quad p_3 = \frac{1}{l}\mathcal{H}_3, \quad etc.$$

Moreover, the function $w(Q) + w(\epsilon_{\pm} Q)$ does not depend on the sheet and it is equal to $2\Delta(\lambda)$. Thus $\Delta(\lambda(Q)) = \cosh ip(Q)2l$ and the formula

$$dp = \pm\frac{1}{i2l} d\cosh^{-1}\Delta(\lambda) = \pm\frac{1}{i2l}\frac{d\Delta(\lambda)}{\sqrt{\Delta^2 - 1}}$$

implies that the differential dp is of the second kind with double poles at the infinities: $\pm dp = d\left(\frac{\lambda}{2} + O(1)\right)$. The formula also implies that the differential dp is pure complex on the real ovals. At the same time, the condition $w(\boldsymbol{\lambda}_k^-) = w(\boldsymbol{\lambda}_k^+)$ requires the increment $p(\boldsymbol{\lambda}_k^+) - p(\boldsymbol{\lambda}_k^-)$ to be real. Therefore,

$$\int_{a_k} dp = 0, \qquad\qquad k = 1, \ldots, g+1.$$

Since the Floquet multiplies are single–valued on Γ, for the b–periods we have

$$\int_{b_k} dp = \frac{\pi n_{b_k}}{l}, \qquad\qquad n_{b_k} \in \mathbb{Z}, \quad k = 1, \ldots, g. \tag{2.12}$$

These are the *periodicity conditions*, [19]. This completes our description of the curve Γ for finite gap potentials. All this can be extended with evident modifications to the general infinite genus case.

Now we consider general smooth potentials from $\mathcal{M}$ and define the 2-vector

$$\mathfrak{e}(x, Q) = \begin{bmatrix} \mathfrak{e}_1(x, Q) \\ \mathfrak{e}_2(x, Q) \end{bmatrix}, \qquad\qquad Q \in \Gamma,$$

to be an eigenvector of the monodromy matrix $M(x, Q)$ corresponding to the eigenvalue $w(Q)$:

$$M(x, Q)\mathfrak{e}(x, Q) = w(Q)\mathfrak{e}(x, Q).$$

Evidently, the components of the vector $\mathfrak{e}$ are defined up to a multiplicative constant, but their ratio is defined uniquely. Let us introduce the Weyl function $\mathcal{X}(x, Q)$ on the Riemann surface Γ by the formula

$$\mathcal{X}(x, Q) = \frac{\mathfrak{e}_2(x, Q)}{\mathfrak{e}_1(x, Q)}. \tag{2.13}$$

We constructed for each x the direct spectral map from the space $\mathcal{M}$ of all smooth periodic potentials to the space of pairs:

$$\mathcal{M} \longrightarrow (\Gamma, \mathcal{X}(x, Q)). \tag{2.14}$$

The direct spectral transform 2.14 is invertible. It follows from the general discussion in [22]. Therefore, for each x there exists an inverse spectral transform

$$(\Gamma, \mathcal{X}(x,Q)) \longrightarrow \mathcal{M}. \tag{2.15}$$

In order to establish analytic properties of the function $\mathcal{X}(x,Q)$ we relate it to the Floquet solutions of the spectral problem 2.4. The Floquet solution is a vector–function

$$\boldsymbol{e}(x,y,Q) = \begin{bmatrix} e_1(x,y,Q) \\ e_2(x,y,Q) \end{bmatrix},$$

which is the solution of the auxiliary spectral problem 2.4 with the property

$$\boldsymbol{e}(x+2l,y,Q) = w(Q)\boldsymbol{e}(x,y,Q) \tag{2.16}$$

for all x. It is easy to check that if this identity holds at some point x_0 then it holds for all x. This condition determines the Floquet solution up to a multiplicative constant. The variable y plays the role of a parameter. The Floquet solution is uniquely determined by the normalization condition

$$e_1(y,y,Q) + e_2(y,y,Q) = 1. \tag{2.17}$$

Evidently,

$$\mathcal{X}(x,Q) = \frac{e_2(x,y,Q)}{e_1(x,y,Q)} \tag{2.18}$$

and the Weyl function is $2l$–periodic $\mathcal{X}(x+2l,Q) = \mathcal{X}(x,Q)$.

Example 2.4. The Floquet solution for the trivial potential $\psi \equiv 0$.

The Floquet solution is given by the formula

$$\boldsymbol{e}(x,y,Q) = e^{+i\frac{\lambda}{2}(x-y)}\boldsymbol{e}_0 = e^{+i\frac{\lambda}{2}(x-y)} \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad Q \in \Gamma_+,$$

$$\boldsymbol{e}(x,y,Q) = e^{-i\frac{\lambda}{2}(x-y)}\hat{\boldsymbol{e}}_0 = e^{-i\frac{\lambda}{2}(x-y)} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad Q \in \Gamma_-.$$

It has no poles in the affine part of the curve. We put $\mathcal{X}(x,Q) = \infty$ when $Q \in \Gamma_+$ and $\mathcal{X}(x,Q) = 0$ when $Q \in \Gamma_-$.

Example 2.5. The one gap potential $\psi(x) = Ce^{i\pi x n/l}$.

In this case the only one gap $\lambda_n^- < \lambda_n^+$ is open and all other are closed. We consider the desingularized curve

$$\hat{\Gamma} = \{\hat{Q} = (\lambda, y) \in \mathbb{C}^2 : \quad y^2 = -(\lambda_n^- - \lambda)(\lambda_n^+ - \lambda)\}.$$

Let

$$\alpha = \frac{\lambda_n^+ + \lambda_n^-}{2}, \qquad \eta = \frac{\lambda_n^+ - \lambda_n^-}{2}.$$

The equation for $\hat{\Gamma}$ in the new notation can be written as $y^2 = \eta^2 - (\alpha - \lambda)^2$. According to Lemma 2, [21], we have

$$\alpha = \frac{\pi n}{l}, \qquad \eta = 2|C|.$$

The rational curve has the representation

$$\begin{aligned} \lambda &= z + \frac{\eta^2}{4(z-\alpha)}, \\ y &= i(z-\alpha) + \frac{\eta^2}{4i(z-\alpha)}, \end{aligned}$$

with the uniformization parameter $z = z[\hat{Q}]$. Evidently $z[\hat{P}_+] = \infty$ and $z[\hat{P}_-] = \alpha$. In the following we identify the points on the curve and their coordinate $z = z(\hat{Q})$.

Let $\epsilon_\pm$ be a holomorphic involution on the curve $\hat{\Gamma}$ permuting sheets:

$$\epsilon_\pm : \ (\lambda, y) \longrightarrow (\lambda, -y).$$

Denoting $\epsilon_\pm z = z(\epsilon_\pm \hat{Q})$ we have

$$\epsilon_\pm z = \alpha + \frac{\eta^2}{4(z-\alpha)}.$$

Let us also define on $\hat{\Gamma}$ an antiholomorphic involution

$$\epsilon_a : (\lambda, y) \longrightarrow (\bar{\lambda}, \bar{y}).$$

For this antiinvolution we have

$$\epsilon_a z = \alpha + \frac{\eta^2}{4(\overline{z}-\alpha)}.$$

Points on the fixed "real" oval are points of the form

$$z = \alpha + \frac{\eta}{2} e^{i\varphi}, \qquad \varphi \in \mathbb{R}^1.$$

The function $p(\hat{Q}) = p(z)$ is single valued and is given by the formula

$$p(z) = \frac{1}{2}\left(z - \frac{\eta^2}{4(z-\alpha)}\right).$$

It is normalized in a such a way that in the expansion at infinity, $p_0^+ = 0$ and $p_0^- = \alpha$. It can be verified directly, that

$$p(z) + p(\epsilon_\pm z) = \alpha.$$

In order to write the Floquet solution let us introduce the image of the pole $\hat{\boldsymbol{\gamma}} = \hat{\boldsymbol{\gamma}}(y)$:

$$z[\hat{\boldsymbol{\gamma}}] = \alpha + z_{\hat{\boldsymbol{\gamma}}} = \alpha + \frac{\eta}{2} e^{i\varphi_{\hat{\boldsymbol{\gamma}}}}.$$

Evidently, it lies on the real oval. We also introduce two functions

$$h_+(z|\hat{\boldsymbol{\gamma}}) = \frac{z-\alpha}{z-\alpha-z_{\hat{\boldsymbol{\gamma}}}}, \qquad h_-(z|\hat{\boldsymbol{\gamma}}) = -\frac{z_{\hat{\boldsymbol{\gamma}}}}{z-\alpha-z_{\hat{\boldsymbol{\gamma}}}}.$$

Evidently,

$$h_+(z|\hat{\boldsymbol{\gamma}}) + h_-(z|\hat{\boldsymbol{\gamma}}) = 1.$$

It can be verified directly

$$h_+(\epsilon_\pm z|\hat{\boldsymbol{\gamma}}) = h_-(z|\epsilon_\pm \hat{\boldsymbol{\gamma}}),$$

and

$$h_-(\epsilon_\pm z|\hat{\boldsymbol{\gamma}}) = h_+(z|\epsilon_\pm \hat{\boldsymbol{\gamma}}).$$

Now we can write an explicit formula for the Floquet solution

$$\boldsymbol{e}(x,y,z) = \boldsymbol{e}(x,y,\hat{Q}) = \begin{bmatrix} h_-(z|\hat{\boldsymbol{\gamma}}(y)) & e^{-i\alpha(x-y)} \\ h_+(z|\hat{\boldsymbol{\gamma}}(y)) & \end{bmatrix} e^{ip(z)(x-y)}.$$

Finally, take $x=y$ in the formula and use the formula for the Floquet solution

$$\mathcal{X}(x,z) = \frac{e_2(x,x,z)}{e_1(x,x,z)} = \frac{h_+(x,z)}{h_-(x,z)} = -\frac{z-\alpha}{z_{\hat{\boldsymbol{\gamma}}(x)}}.$$

We see that $\mathcal{X}(x,\hat{Q})$ has a fixed pole at $z=\infty$ which corresponds to $\hat{P}_+$. It also has a fixed zero at $z=\alpha$ which corresponds to $\hat{P}_-$. This observation completes our considerations of the rational case.

For a general finite gap potential the situation is more complicated. We will show that the Floquet solution is given by the explicit formulas 2.20-2.21. Now we will state the properties of the Floquet solutions.

Lemma 2.6. *The Floquet solution satisfies the identity*

$$e(x,y,\epsilon_a Q) = \sigma_1 \overline{\boldsymbol{e}(x,y,Q)}. \tag{2.19}$$

The Floquet solution $\boldsymbol{e}(x,y,Q)$ has poles common for both components at the points

$$\boldsymbol{\gamma}_1(y), \boldsymbol{\gamma}_2(y), \ldots, \boldsymbol{\gamma}_{g+1}(y).$$

The poles depend on the normalization point y. The projections of poles $\mu_k(y) = \lambda(\boldsymbol{\gamma}_k(y))$ are real. Each $\boldsymbol{\gamma}_k(y)$ lies on the real oval above the corresponding open gap $[\lambda_k^-, \lambda_k^+]$. The first component $e_1(x,Q)$ has $g+1$ zeros at the points

$$\boldsymbol{\sigma}_1(x),\ \boldsymbol{\sigma}_2(x), \ldots, \boldsymbol{\sigma}_g(x), \boldsymbol{\sigma}_{g+1} = P_+.$$

The second component $e_2(x,Q)$ has $g+1$ zeros at the points

$$\boldsymbol{\sigma}'_1(x),\ \boldsymbol{\sigma}'_2(x), \ldots, \boldsymbol{\sigma}'_g(x), \boldsymbol{\sigma}'_{g+1} = P_-.$$

For both components the first g zeros depend on the parameter x. In the vicinity of infinities the function $\boldsymbol{e}(x,y,Q)$ has the asymptotic behavior

$$\boldsymbol{e}(x,y,Q) = e^{\pm i\frac{\lambda}{2}(x-y)}\left[\boldsymbol{e}_0/\hat{\boldsymbol{e}}_0 + o(1)\right], \qquad Q \in (P_\pm).$$

Proof.[7] The proof is based on the explicit formula for the Floquet solution $\boldsymbol{e}(x,y,Q)$:

$$\boldsymbol{e}(x,y,Q) = A(y,Q)\left[\begin{array}{c} m_{11} \\ m_{21} \end{array}\right](x,y,\lambda) + (1-A(y,Q))\left[\begin{array}{c} m_{12} \\ m_{22} \end{array}\right](x,y,\lambda), \quad (2.20)$$

where $\lambda = \lambda(Q)$ and the first component $A(y,Q)$ is

$$A(y,Q) = \frac{M_{12}}{M_{12} - M_{11} + w(Q)}(y,\lambda) = \frac{w(Q) - M_{22}}{M_{21} - M_{22} + w(Q)}(y,\lambda). \qquad (2.21)$$

□

The Floquet solution $\boldsymbol{e}(x,y,Q)$ near infinities can be expanded into the asymptotic series

$$\boldsymbol{e}(x,y,Q) = e^{+i\frac{\lambda}{2}(x-y)} \sum_{s=0}^{\infty} \boldsymbol{e}_s(x,y)\lambda^{-s} = e^{+i\frac{\lambda}{2}(x-y)} \sum_{s=0}^{\infty} \left[\begin{array}{c} b_s \\ d_s \end{array}\right]\lambda^{-s}, \quad Q \in (P_+),$$

$$\boldsymbol{e}(x,y,Q) = e^{-i\frac{\lambda}{2}(x-y)} \sum_{s=0}^{\infty} \hat{\boldsymbol{e}}_s(x,y)\lambda^{-s} = e^{-i\frac{\lambda}{2}(x-y)} \sum_{s=0}^{\infty} \left[\begin{array}{c} \overline{d}_s \\ \overline{b}_s \end{array}\right]\lambda^{-s}, \quad Q \in (P_-),$$

with

$$b_0 = 0, \qquad d_0 = 1,$$

and

$$b_1 = -i\overline{\psi}(x), \qquad d_1 = i\overline{\psi}(y) - i\int_y^x |\psi|^2 dx'.$$

Since ϵ_a permutes infinities the formula 2.19 implies that the asymptotic expansions are connected.

Now take $x = y$ in formula 2.18 and use formulas 2.20–2.21. For the Weyl function we have

$$\mathcal{X}(x,Q) = \frac{1 - A(x,Q)}{A(x,Q)} = \frac{w(Q) - M_{11}(x,\lambda)}{M_{12}(x,\lambda)} = \frac{M_{21}(x,\lambda)}{w(Q) - M_{22}(x,\lambda)}. \qquad (2.22)$$

Lemma 2.6 implies the following analytic properties of the Weyl function $\mathcal{X}(x,Q)$ for finite gap potentials

- For a fixed x the function $\mathcal{X}(x,Q)$ is meromorphic on Γ and satisfies the identity

$$\mathcal{X}(x,\epsilon_a Q) = \frac{1}{\overline{\mathcal{X}(x,Q)}}. \qquad (2.23)$$

  This implies that on the real ovals $\mathcal{X}$ takes values on the unit circle.
- It has g poles $\boldsymbol{\sigma}_k(x)$, $k = 1,\ldots,g$, and zeros $\boldsymbol{\sigma}'_k(x)$, $k = 1,\ldots,g$. These poles and zeros depend on the parameter x.

[7] The complete proof of the Lemma with $y = -l$ can be found in [25].

- The function $\mathcal{X}$ has a simple pole at $\boldsymbol{\sigma}_{g+1} = P_+$ and a zero at $\boldsymbol{\sigma}'_{g+1} = P_-$. In the vicinity of these points it has the asymptotic expansion

$$\mathcal{X}(x,Q) = \frac{\lambda}{-i\overline{\psi}(x)} + O(1), \qquad Q \in (P_+); \tag{2.24}$$

$$\mathcal{X}(x,Q) = \frac{i\psi(x)}{\lambda} + O\left(\frac{1}{\lambda^2}\right), \qquad Q \in (P_-). \tag{2.25}$$

Properties 2.23 and 2.24–2.25 are true for all potentials from the phase space $\mathcal{M}$.

Remark 2.7. In the finite gap case the proof of injectivity for the map 2.14 can be obtained by pure algebraic methods. Indeed when the function $\mathcal{X}(x,Q)$ is known for some fixed value of x, then two sets of zeros $\boldsymbol{\sigma}_k(x),\ k = 1,\dots,g+1$, and $\boldsymbol{\sigma}'_k(x),\ k = 1,\dots,g+1$, are known for each component of the Floquet solution. The components are the Baker–Akhiezer function on Γ, [12]. They are determined uniquely by the standard asymptotic at infinities and the set of poles or zeros. Thus the pair $(\Gamma, \mathcal{X}(x,Q))$ determines the Baker-Akhiezer function[8] $\boldsymbol{e}(\bullet, x, Q)$ and correspondingly the potential $\psi(\bullet)$.

Remark 2.8. We constructed for each x the direct spectral map from the space of $g+1$ gap potentials to the space of pairs $(\Gamma, \mathcal{X}(x,Q))$. The target space has dimension $2g+2$. Indeed the spectral curve Γ is specified by $2g+2$ real branch points. The periodicity conditions 2.12 cut g real degrees of freedom. Also on such a curve there exists a distinguished differential $dp(Q)$ of the second kind. The differential $dp(Q)$ determines the multivalued function $p(Q)$. The coefficient $p_0^{\pm}$ of its asymptotic expansion has the form 2.11 and determines the branch points up to a discrete set of shifts. Therefore, the spectral curve Γ is specified by $g+1$ real parameters. The function $\boldsymbol{e}(\bullet, x, Q)$ is uniquely determined by the fixed asymptotic at infinities and the poles $\boldsymbol{\gamma}_k(x)$. These poles parametrize the set of all meromorphic functions $\mathcal{X}(x,Q)$. Therefore the set of functions $\mathcal{X}(x,Q)$ for fixed x is topologicaly equivalent to $g+1$ dimensional real torus formed by the real ovals $a_1 \times \ldots \times a_{g+1}$ (see Figure 5).

The space of all smooth periodic potentials is stratified manifold. Each strata has an even dimension (possibly infinite).

Remark 2.9. In example 2.5 we considered the simplest case of a rational curve. In this case the only zero and pole stay fixed and do not depend on the parameter x at all.

For $g > 0$ when x changes over a period, the pole $\boldsymbol{\sigma}_k(x),\ k = 1,\dots,g$, makes a closed loop on the surface Γ. We denote this loop by $s_k = \{\boldsymbol{\sigma}_k(x), x \in \mathbb{R}^1\}$. The zeros $\boldsymbol{\sigma}'_k(x),\ k = 1,\dots,g$, of the function $\mathcal{X}(x,Q)$ have the same property. We denote the corresponding loops by $s'_k = \{\boldsymbol{\sigma}'_k(x), x \in \mathbb{R}^1\}$. Due to 2.23 we have

[8] $\bullet$ here signifies the argument of the function.

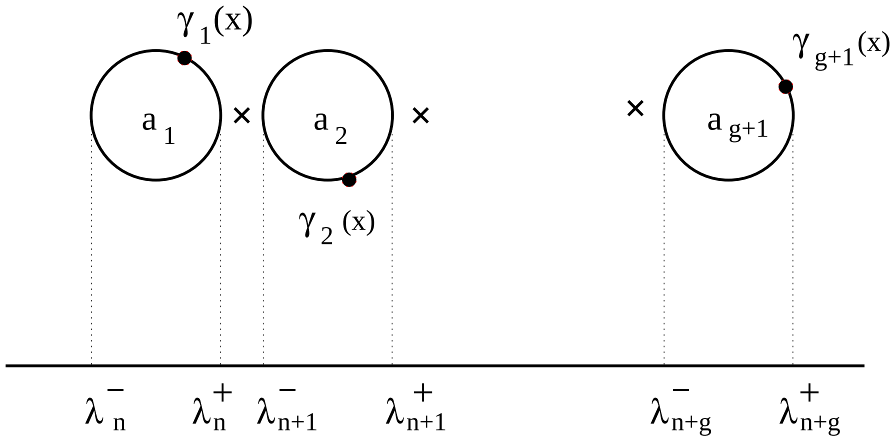

FIGURE 5. The $g+1$ dimensional torus of potentials.

$\epsilon_a \boldsymbol{\sigma}_k(x) = \boldsymbol{\sigma}'_k(x)$. The topology of the set of zeros is an interesting question which we do not address here.

2.3. **Analytic properties of the Floquet multipliers.** In this section we explain the relation between the Floquet solutions and more general, so called, Weyl solutions.

Let $\boldsymbol{f}^T$ denote a transposition of vector $\boldsymbol{f}$ and let $\boldsymbol{f}^*$ denote the adjoint of the vector $\boldsymbol{f}$; $\boldsymbol{L}^2(a,b)$ be a space of vector functions with the property

$$\int_a^b \boldsymbol{f}^*(x,\lambda)\boldsymbol{f}(x,\lambda)\,dx < \infty.$$

For a periodic potential, the quadratic equation 2.7 implies that for each value λ_0 of the spectral parameter we have $w(Q)w(\epsilon_\pm Q)) = 1$, where $\lambda(Q) = \lambda_0$. Therefore, either

$$|w(Q))| < 1, \qquad |w(\epsilon_\pm Q))| > 1;$$

or

$$|w(Q))| = |w(\epsilon_\pm Q))| = 1.$$

The set of all $\lambda \in \mathbb{C}$, where the first/second possibility occurs is called the instability/stability area. For an unstable λ the Floquet solution corresponding to the sheet where $|w(Q))| < 1$ grows exponentially when $x \to -\infty$, while the Floquet solution on another sheet with $|w(\epsilon_\pm Q))| > 1$ grows exponentially when $x \to +\infty$. For a stable λ both Floquet solutions stay bounded for all values of x. Stability of solutions of periodic (in the x-variable) systems is a classical subject, see [11, 9]. We need just some elementary facts of this theory.

If the potential vanishes identically (see Example 2.2), then all λ with $\Im\lambda \neq 0$ form the instability area. The real λ belong to the stability area. For a finite gap

potential all λ with $\Im\lambda \neq 0$ remain unstable, but all open gaps on the real line belong to the instability area (Figure 6).

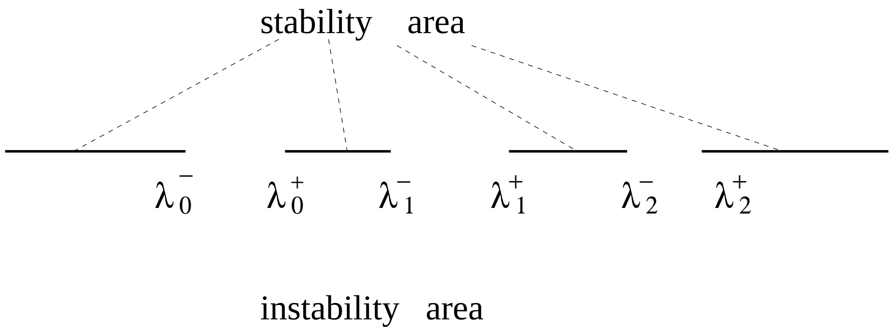

FIGURE 6. Stability areas on the λ-plane for 3 gap potential .

The Weyl solution $\boldsymbol{f}(x,\lambda)$ of the eigenvalue problem 2.4 with arbitrary[9] potential and λ such that $\Im\lambda \neq 0$ is defined by the property that it belongs to $\boldsymbol{L}^2[0,+\infty)$ or $\boldsymbol{L}^2(-\infty,0]$, [27]. Evidently, any Weyl solution is defined up to a multiplicative constant. For the Dirac operator such solutions exist and are unique under the sole assumption of continuity of the potential, [14]. For a finite-gap potential, if $Q \in \Gamma$ is such that $\Im\lambda(Q) \neq 0$, then $|w(Q)| < 1$ or $|w(Q)| > 1$ and the Floquet solution $\boldsymbol{e}(x,y,Q)$ belongs to $\boldsymbol{L}^2[0,+\infty)$ or $\boldsymbol{L}^2(-\infty,0]$ correspondingly. Therefore, the Floquet solution is a particular case of the Weyl solution.

In the case of a periodic potential, the hyperelliptic curve $\hat{\Gamma}$ can be obtained by compactification of the spectral cover, supporting the Weyl solutions, see [22]. Indeed in this case one can take two copies of $\mathbb{CP}^1$ and cut them along the open gaps. Then identifying edges, one obtains the Riemann surface of genus g equal to one less than the number of open gaps. The $2g+2$ points of the simple spectrum become fixed points of the holomorphic involution permuting sheets of the curve. Thus the compactified cover is the hyperelliptic surface $\hat{\Gamma}$. In the previous section we defined this surface by an algebraic equation.

Remark 2.10. The characterization of all *finite gap potentilas* of the Dirac operator was obtained along these lines in the paper [4]. In their arguments de Concini and Jonhson analytically extend the functions originally defined on the spectral cover to the functions on smooth hyperelliptic curve.

2.4. **The linear fractional transformations of the Weyl function.** Let $\boldsymbol{f}(x,\lambda)$ be some Weyl solution of the auxiliary spectral problem 2.4 corresponding to some fixed value of the spectral parameter. For any x the function $\boldsymbol{f}(x,\lambda)$ is a vector from $\mathbb{C}^2$. It can be represented in the form

$$\boldsymbol{f}(x,\lambda) = c^1(x)\boldsymbol{k}_1 + c^2(x)\boldsymbol{k}_2,$$

[9] not necessarily periodic.

where

$$\boldsymbol{k}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \boldsymbol{k}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The quantities $c^1(x)$ and $c^2(x)$ are defined up to a multiplicative constant, but their ratio

$$\mathcal{X}(x) = \frac{c^2(x)}{c^1(x)}$$

is defined uniquely. Consider $\tilde{\boldsymbol{k}}_1, \tilde{\boldsymbol{k}}_2$, some other bases of $\mathbb{C}^2$. Then

$$\boldsymbol{f}(x,\lambda) = \tilde{c}^1(x)\tilde{\boldsymbol{k}}_1 + \tilde{c}^2(x)\tilde{\boldsymbol{k}}_2,$$

and the function $\tilde{\mathcal{X}}(x)$ can be defined as before:

$$\tilde{\mathcal{X}}(x) = \frac{\tilde{c}^2(x)}{\tilde{c}^1(x)}.$$

If the two bases are connected by the formula

$$\begin{aligned} \boldsymbol{k}_1 &= a\tilde{\boldsymbol{k}}_1 + b\tilde{\boldsymbol{k}}_2, \\ \boldsymbol{k}_2 &= c\tilde{\boldsymbol{k}}_1 + d\tilde{\boldsymbol{k}}_2; \end{aligned}$$

where a, b, c and d are complex constants, then the function $\tilde{\mathcal{X}}(x)$ and $\mathcal{X}(x)$ are connected by the linear fractional transformation

$$\tilde{\mathcal{X}} = \frac{d\mathcal{X} + b}{c\mathcal{X} + a}. \tag{2.26}$$

Our definition of the Weyl function $\mathcal{X}(x,Q)$ in formula 2.18 corresponds to the bases $\boldsymbol{k}_1, \boldsymbol{k}_2$ in $\mathbb{C}^2$. Any other bases $\tilde{\boldsymbol{k}}_1, \tilde{\boldsymbol{k}}_2$ will lead to a new Weyl type function $\tilde{\mathcal{X}}(x,Q)$ connected to $\mathcal{X}(x,Q)$ by formula 2.26.

Geometrically the transformation 2.26 changes the global uniformization parameter in the target space for the map

$$\mathcal{X}(x,Q) : \Gamma \longrightarrow \mathbb{CP}^1.$$

3. The Poisson bracket.

3.1. **The Poisson bracket for the Weyl function.** In this section we compute the image of the Poisson bracket 2.2 under the direct spectral transform 2.14.

Let us introduce the function

$$\Omega(Q) = \frac{w^2(Q)+1}{w^2(Q)-1} = \coth ip(Q)2l.$$

It is easy to see that

$$\Omega(\epsilon_\pm Q) = -\Omega(Q),$$

and

$$\Omega(\epsilon_a Q) = \overline{\Omega}(Q).$$

In the finite part of the curve there are two types of important points. These are singular points and branch points. The function $\lambda(Q)$ where $Q = (\lambda, w) \in \Gamma$ is a local parameter everywhere on the curve with the exception of these points. At the branch and singular points the function $\Omega(Q)$ has poles because $w^2(Q) = 1$. The zeros of the function coincide with the points of the anti-periodic spectrum of the problem on the doubled interval $2 \times 2l$.

Now we are ready to state the main result of this section.

Theorem 3.1. *Let $Q = (\lambda, w(Q))$ and $P = (\mu, w(P))$ be different from the branch and singular points of Γ and poles of $\mathcal{X}$. The bracket 2.2 for two functions $\mathcal{X}(Q) = \mathcal{X}(x, Q)$ and $\mathcal{X}(P) = \mathcal{X}(y, P)$ is given by the formula*

$$\{\mathcal{X}(Q), \mathcal{X}(P)\} = -2 \times \frac{(\mathcal{X}(Q) - \mathcal{X}(P))^2}{\lambda - \mu} \times \frac{\Omega(Q) + \Omega(P)}{2}. \tag{3.1}$$

First for the proof we need some identity for the quartic products of solutions of the auxiliary spectral problem.

Lemma 3.2. *Let the column vectors $\boldsymbol{g}^+(x, \mu)$, $\boldsymbol{f}^+(x, \lambda)$ satisfy*

$$\boldsymbol{g}^{+\prime}(x, \mu) = V(x, \mu)\boldsymbol{g}^+(x, \mu), \qquad \boldsymbol{f}^{+\prime}(x, \lambda) = V(x, \lambda)\boldsymbol{f}^+(x, \lambda);$$

and the row vectors $\boldsymbol{g}^-(x, \mu)$, $\boldsymbol{f}^-(x, \lambda)$ satisfy

$$\boldsymbol{g}^{-\prime}(x, \mu) = -\boldsymbol{g}^-(x, \mu)V(x, \mu), \qquad \boldsymbol{f}^{-\prime}(x, \lambda) = -\boldsymbol{f}^-(x, \lambda)V(x, \lambda).$$

The following identity holds:

$$f_2^+ f_1^- g_1^+ g_2^- - f_1^+ f_2^- g_2^+ g_1^- = \frac{1}{i(\lambda - \mu)} \frac{d}{dx} \left[(\boldsymbol{f}^- I \boldsymbol{g}^+)(\boldsymbol{g}^- I \boldsymbol{f}^+) \right]. \tag{3.2}$$

Proof. The identity can be verified by differentiation. □

The next lemma provides the bracket for entries of the transition matrix.

Lemma 3.3. *The following formulas hold for the entries $m_{ij}(\lambda) = m_{ij}(x, y, \lambda)$, $x > y$, of the transition matrix $M(x, y, \lambda)$*

$$\{m_{11}(\lambda), m_{12}(\mu)\} = K \times \frac{m_{12}(\lambda)m_{11}(\mu) - m_{12}(\mu)m_{11}(\lambda)}{\lambda - \mu} \tag{3.3}$$

$$\{m_{11}(\lambda), m_{21}(\mu)\} = K \times \frac{m_{11}(\lambda)m_{21}(\mu) - m_{11}(\mu)m_{21}(\lambda)}{\lambda - \mu} \tag{3.4}$$

$$\{m_{11}(\lambda), m_{22}(\mu)\} = K \times \frac{m_{12}(\lambda)m_{21}(\mu) - m_{12}(\mu)m_{21}(\lambda)}{\lambda - \mu} \tag{3.5}$$

$$\{m_{12}(\lambda), m_{21}(\mu)\} = K \times \frac{m_{11}(\lambda)m_{22}(\mu) - m_{11}(\mu)m_{22}(\lambda)}{\lambda - \mu} \tag{3.6}$$

$$\{m_{12}(\lambda), m_{22}(\mu)\} = K \times \frac{m_{12}(\lambda)m_{22}(\mu) - m_{12}(\mu)m_{22}(\lambda)}{\lambda - \mu} \tag{3.7}$$

$$\{m_{21}(\lambda), m_{22}(\mu)\} = K \times \frac{m_{22}(\lambda)m_{21}(\mu) - m_{22}(\mu)m_{21}(\lambda)}{\lambda - \mu}. \tag{3.8}$$

with $K = -2$. All other brackets vanish.

Proof. We will prove the first identity. The other formulas can be proved along the same lines.

Let $M^\bullet = \delta M$ be a variation of $M(x, y, \lambda)$ in response to the variation of $\psi(z)$ and $\overline{\psi}(z)$, $y \leq z \leq x$. Then $M^{\bullet\prime} = VM^\bullet + V^\bullet M$. The solution of this nonhomogenious equation is

$$M^\bullet(x, y, \lambda) = M(x, y, \lambda) \int\limits_0^x M^{-1}(\xi, y, \lambda)V^\bullet(\xi)M(\xi, y, \lambda)\, d\xi$$

Therefore

$$\frac{\delta M(x, y, \lambda)}{\delta\psi(z)} = M(x, z, \lambda) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} M(z, y, \lambda),$$

$$\frac{\delta M(x, y, \lambda)}{\delta\overline{\psi}(z)} = M(x, z, \lambda) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} M(z, y, \lambda).$$

Using the formulas for the gradients, we have

$$\begin{aligned}\{m_{11}(\lambda), m_{12}(\mu)\} = 2i \int_y^x dz\, [&m_{11}(x, z, \lambda)m_{21}(z, y, \lambda)m_{21}(x, z, \mu)m_{12}(z, y, \mu) - \\ &- m_{12}(x, z, \lambda)m_{11}(z, y, \lambda)m_{11}(x, z, \mu)m_{22}(z, y, \mu)].\end{aligned}$$

Since the matrix $M(x, y, \lambda)$ satisfies the differential equations

$$\frac{\partial M(x, y, \lambda)}{\partial x} = V(x, \lambda)M(x, y, \lambda), \qquad \frac{\partial M(x, y, \lambda)}{\partial y} = -M(x, y, \lambda)V(y, \lambda),$$

we can apply the identity of Lemma 3.2 to the computation of the integral. If we put

$$\boldsymbol{f}^{+} = \begin{bmatrix} m_{11} \\ m_{21} \end{bmatrix} (z,y,\lambda), \qquad \boldsymbol{g}^{+} = \begin{bmatrix} m_{12} \\ m_{22} \end{bmatrix} (z,y,\mu);$$

and

$$\boldsymbol{f}^{-} = [m_{11}, m_{12}](x,z,\lambda), \qquad \boldsymbol{g}^{-} = [m_{11}, m_{12}](x,z,\mu);$$

then we can apply the identity in the variable z:

$$\{m_{11}(\lambda), m_{12}(\mu)\} = \frac{2}{\lambda-\mu}\left[(\boldsymbol{f}^{-}I\boldsymbol{g}^{+})(\boldsymbol{g}^{-}I\boldsymbol{f}^{+})\right]|_{y}^{x}.$$

Using boundary conditions we obtain the result. $\square$

Now we are ready to prove Theorem 3.1.

Proof. For the function $\mathcal{X}(x,Q)$ we use representation 2.22. We use the result of Lemma 3.3 with an unspecified value of the constant K. We split the computation of the Poisson bracket

$$\begin{aligned}
\{\mathcal{X}(Q), \mathcal{X}(P)\} &= \{\frac{w(Q)-M_{11}(\lambda)}{M_{12}(\lambda)}, \frac{w(P)-M_{11}(\mu)}{M_{12}(\mu)}\} \\
&= \{w(Q)-M_{11}(\lambda), w(P)-M_{11}(\mu)\}\frac{1}{M_{12}(\lambda)}\frac{1}{M_{12}(\mu)} \qquad (3.9)\\
&+ \{w(Q)-M_{11}(\lambda), \frac{1}{M_{12}(\mu)}\}\frac{1}{M_{12}(\lambda)}(w(P)-M_{11}(\mu)) \qquad (3.10)\\
&+ \{\frac{1}{M_{12}(\lambda)}, w(P)-M_{11}(\mu)\}(w(Q)-M_{11}(\lambda))\frac{1}{M_{12}(\mu)} \qquad (3.11)\\
&+ \{\frac{1}{M_{12}(\lambda)}, \frac{1}{M_{12}(\mu)}\}(w(Q)-M_{11}(\lambda))(w(P)-M_{11}(\mu)).
\end{aligned}$$

into small steps. First note that due to Lemma 3.3 the last line vanishes.

Step 1. Using the identities

$$\{w(Q), M_{11}(\mu)\} = K \times \frac{1}{2}\frac{\partial w}{\partial \Delta}(Q)\frac{M_{21}(\lambda)M_{12}(\mu)-M_{21}(\mu)M_{12}(\lambda)}{\lambda-\mu},$$

$$\{w(P), M_{11}(\lambda)\} = K \times \frac{1}{2}\frac{\partial w}{\partial \Delta}(P)\frac{M_{21}(\lambda)M_{12}(\mu)-M_{21}(\mu)M_{12}(\lambda)}{\lambda-\mu};$$

which follow from Lemma 3.3 for 3.9, we have

$$\begin{aligned}
\{w(Q) \quad &- \quad M_{11}(\lambda), w(P)-M_{11}(\mu)\}\frac{1}{M_{12}(\lambda)}\frac{1}{M_{12}(\mu)} = \\
= \qquad & \frac{K}{\lambda-\mu} \times \frac{1}{M_{12}(\lambda)M_{12}(\mu)}\frac{M_{21}(\lambda)M_{12}(\mu)-M_{21}(\mu)M_{12}(\lambda)}{2}\left(\frac{\partial w}{\partial \Delta}(P) - \frac{\partial w}{\partial \Delta}(Q)\right).
\end{aligned}$$

Introducing the function $\mathcal{D}(\lambda) = \frac{1}{2}(M_{11}(\lambda) - M_{22}(\lambda))$ we have

$$M_{22}(\lambda) = \Delta(\lambda) - \mathcal{D}(\lambda).$$

This together with formula 2.22 imply

$$M_{21}(\lambda) = \mathcal{X}(Q)(w(Q) + \mathcal{D}(\lambda) - \Delta(\lambda)).$$

After simple transformations, 3.9 becomes

$$K \times \frac{1}{(\lambda-\mu)}\left[\frac{1}{M_{12}(\lambda)}\mathcal{X}(Q)\frac{w(Q)+\mathcal{D}(\lambda)-\Delta(\lambda)}{2}\left(\frac{\partial w}{\partial \Delta}(P) - \frac{\partial w}{\partial \Delta}(Q)\right) - \right. \tag{3.12}$$

$$\left. - \frac{1}{M_{12}(\mu)}\mathcal{X}(P)\frac{w(P)+\mathcal{D}(\mu)-\Delta(\mu)}{2}\left(\frac{\partial w}{\partial \Delta}(P) - \frac{\partial w}{\partial \Delta}(Q)\right)\right]. \tag{3.13}$$

Step 2. The second line 3.10 is equal to

$$-\left(\{w(Q), M_{12}(\mu)\} - \{M_{11}(\lambda), M_{12}(\mu)\}\right)\frac{\mathcal{X}(P)}{M_{12}(\lambda)M_{12}(\mu)}.$$

The third line 3.11 is equal to

$$\left(\{w(P), M_{12}(\lambda)\} - \{M_{11}(\mu), M_{12}(\lambda)\}\right)\frac{\mathcal{X}(Q)}{M_{12}(\lambda)M_{12}(\mu)}.$$

After simple transformations the sum of 3.10-3.11 becomes

$$= \frac{1}{M_{12}(\lambda)M_{12}(\mu)}\left(\mathcal{X}(Q)\{w(P), M_{12}(\lambda)\} - \mathcal{X}(P)\{w(Q), M_{12}(\mu)\}\right) + \tag{3.14}$$

$$+ \frac{1}{M_{12}(\lambda)M_{12}(\mu)}\left(\mathcal{X}(P)\{M_{11}(\lambda), M_{12}(\mu)\} - \mathcal{X}(Q)\{M_{11}(\mu), M_{12}(\lambda)\}\right). \tag{3.15}$$

Step 3. This step consists of simple manipulations with 3.14-3.15. Using the identities

$$\{w(P), M_{12}(\lambda)\} = K \times -\frac{\partial w}{\partial \Delta}(P)\frac{\mathcal{D}(\lambda)M_{12}(\mu) - \mathcal{D}(\mu)M_{12}(\lambda)}{\lambda - \mu},$$

$$\{w(Q), M_{12}(\mu)\} = K \times -\frac{\partial w}{\partial \Delta}(Q)\frac{\mathcal{D}(\lambda)M_{12}(\mu) - \mathcal{D}(\mu)M_{12}(\lambda)}{\lambda - \mu},$$

which follow from Lemma 3.3, we have for 3.14

$$K \times \frac{1}{(\lambda-\mu)}\left[\frac{1}{M_{12}(\lambda)}\left(\mathcal{X}(P)\frac{\partial w}{\partial \Delta}(Q)\mathcal{D}(\lambda) - \mathcal{X}(Q)\frac{\partial w}{\partial \Delta}(P)\mathcal{D}(\lambda)\right) + \right. \tag{3.16}$$

$$\left. + \frac{1}{M_{12}(\mu)}\left(\mathcal{X}(Q)\frac{\partial w}{\partial \Delta}(P)\mathcal{D}(\mu) - \mathcal{X}(P)\frac{\partial w}{\partial \Delta}(Q)\mathcal{D}(\mu)\right)\right]. \tag{3.17}$$

For 3.15 using Lemma 3.3 and

$$M_{11}(\lambda) = \Delta(\lambda) + \mathcal{D}(\lambda),$$

after simple algebra we obtain

$$K \times \frac{1}{(\lambda-\mu)}\left[\frac{1}{M_{12}(\lambda)}(\Delta(\lambda)+\mathcal{D}(\lambda))(\mathcal{X}(Q)-\mathcal{X}(P)) - \right. \tag{3.18}$$

$$\left. - \frac{1}{M_{12}(\mu)}(\Delta(\mu)+\mathcal{D}(\mu))(\mathcal{X}(Q)-\mathcal{X}(P))\right]. \tag{3.19}$$

Step 4. Collecting terms 3.12, 3.16 and 3.18 containing $M_{12}(\lambda)$ in the denominator we have

$$\begin{aligned} K \times \frac{1}{(\lambda-\mu)}\frac{\mathcal{X}(Q)}{M_{12}(\lambda)} &\times \left[\frac{w(Q)+\mathcal{D}(\lambda)-\Delta(\lambda)}{2}\left(\frac{\partial w}{\partial\Delta}(P)-\frac{\partial w}{\partial\Delta}(Q)\right) - \right. \\ &- \left. \mathcal{D}(\lambda)\frac{\partial w}{\partial\Delta}(P)+\Delta(\lambda)+\mathcal{D}(\lambda)\right] - \\ K \times \frac{1}{(\lambda-\mu)}\frac{\mathcal{X}(P)}{M_{12}(\lambda)} &\times \left[\mathcal{D}(\lambda)\frac{\partial w}{\partial\Delta}(Q)-\Delta(\lambda)-\mathcal{D}(\lambda)\right]. \end{aligned}$$

For the first term we have

$$\begin{aligned} \frac{\mathcal{X}(Q)}{M_{12}(\lambda)} &\times \left[\frac{w(Q)+\mathcal{D}(\lambda)-\Delta(\lambda)}{2}\left(\frac{\partial w}{\partial\Delta}(P)-\frac{\partial w}{\partial\Delta}(Q)\right)-\mathcal{D}(\lambda)\frac{\partial w}{\partial\Delta}(P)+\Delta(\lambda)+\mathcal{D}(\lambda)\right] \\ &= \frac{\mathcal{X}(Q)}{M_{12}(\lambda)} \times \left[\frac{w(Q)-\mathcal{D}(\lambda)-\Delta(\lambda)}{2}\frac{\partial w}{\partial\Delta}(P)-(w(Q)-\mathcal{D}(\lambda)-\Delta(\lambda))\right] + \\ &+ \frac{\mathcal{X}(Q)}{M_{12}(\lambda)} \times \left[-\frac{w(Q)+\mathcal{D}(\lambda)-\Delta(\lambda)}{2}\frac{\partial w}{\partial\Delta}(Q)+w(Q)\right]. \end{aligned}$$

The derivative $\partial w/\partial\Delta(Q)$ can be easily computed from the quadratic equation 2.7:

$$\frac{\partial w}{\partial\Delta}(Q) = \frac{w(Q)}{w(Q)-\Delta(\lambda)} = \frac{2w^2(Q)}{w^2(Q)-1}.$$

Therefore

$$\begin{aligned} \mathcal{X}^2(Q)\left[\frac{1}{2}\frac{\partial w}{\partial\Delta}(P)-1\right] &+ \frac{\mathcal{X}(Q)}{M_{12}(\lambda)}\frac{\partial w}{\partial\Delta}(Q)\left[-\frac{w(Q)+\mathcal{D}(\lambda)-\Delta(\lambda)}{2}+w(Q)-\Delta(\lambda)\right] \\ &= \mathcal{X}^2(Q)\left[\frac{1}{2}\frac{\partial w}{\partial\Delta}(P)-1\right]+\frac{\mathcal{X}^2(Q)}{2}\frac{\partial w}{\partial\Delta}(Q). \end{aligned}$$

Note

$$\Omega(Q) = \frac{\partial w}{\partial\Delta}(Q) - 1. \tag{3.20}$$

Therefore, for the first term we obtain

$$\mathcal{X}^2(Q)\frac{\Omega(Q)+\Omega(P)}{2}. \tag{3.21}$$

For the second term we have

$$\begin{aligned}
\frac{\mathcal{X}(P)}{M_{12}(\lambda)} &\times \left[\mathcal{D}(\lambda)\frac{\partial w}{\partial\Delta}(Q)-\Delta(\lambda)-\mathcal{D}(\lambda)\right]= \\
&= \frac{\mathcal{X}(P)}{M_{12}(\lambda)}\times\left[\mathcal{D}(\lambda)\frac{\partial w}{\partial\Delta}(Q)-w(Q)\right]+\frac{\mathcal{X}(P)}{M_{12}(\lambda)}\times[w(Q)-\Delta(\lambda)-\mathcal{D}(\lambda)]= \\
&= -\mathcal{X}(Q)\mathcal{X}(P)\frac{\partial w}{\partial\Delta}(Q)+\mathcal{X}(Q)\mathcal{X}(P).
\end{aligned}$$

Using 3.20 we have

$$-\mathcal{X}(Q)\mathcal{X}(P)\Omega(Q). \tag{3.22}$$

Taking the sum of 3.21 and 3.22 we finally obtain

$$K\times\frac{1}{\lambda-\mu}\left[\mathcal{X}^2(Q)\frac{\Omega(Q)+\Omega(P)}{2}-\mathcal{X}(Q)\mathcal{X}(P)\Omega(Q)\right]. \tag{3.23}$$

Step 5. Collecting terms 3.13, 3.17 and 3.19 containing $M_{12}(\mu)$ in the denominator we have

$$\begin{aligned}
K\times\frac{1}{\lambda-\mu}\frac{\mathcal{X}(P)}{M_{12}(\mu)} &\times \left[\frac{w(P)+\mathcal{D}(\mu)-\Delta(\mu)}{2}\left(\frac{\partial w}{\partial\Delta}(Q)-\frac{\partial w}{\partial\Delta}(P)\right)-\right. \\
&- \left.\mathcal{D}(\mu)\frac{\partial w}{\partial\Delta}(Q)+\Delta(\mu)+\mathcal{D}(\mu)\right]- \\
K\times\frac{1}{\lambda-\mu}\frac{\mathcal{X}(Q)}{M_{12}(\mu)} &\times \left[\mathcal{D}(\mu)\frac{\partial w}{\partial\Delta}(P)-\Delta(\mu)-\mathcal{D}(\mu)\right].
\end{aligned}$$

After we interchange $\lambda\leftrightarrow\mu$ and $Q\leftrightarrow P$ this expression becomes identical to the formula of Step 4. Therefore, it is equal to

$$K\times\frac{1}{\lambda-\mu}\left[\mathcal{X}^2(P)\frac{\Omega(Q)+\Omega(P)}{2}-\mathcal{X}(Q)\mathcal{X}(P)\Omega(P)\right] \tag{3.24}$$

Taking the sum of 3.23 and 3.24 we obtain formula 3.1. □

Remark 3.4. It is interesting to note that the bracket of two meromorphic functions on the Riemann surface is not a meromorphic function in any reasonable sense. The presence of infinitely many intersection or branch points makes it possible to obtain any limiting value for the bracket, when one of the points, say Q, is fixed and P tends to infinity. This is due to the fact that $\mathcal{X}(P)$ viewed as Hamiltonian can open gaps arbitrarily far, and produce a change of topology of the curve.[10]

Remark 3.5. The formula 3.1 formally contains all three cases of Theorem 4.1 in [22]. One has to replace Ω on $+1$ or -1 to get the formulas obtained there.

[10] I indebted to I. Krichever for this remark.

Remark 3.6. We call the formula 3.1 the deformed Atiyah-Hitchin bracket. When the points Q, P approach infinities from imaginary directions (see Remark 2.3) then $w(Q)$ and $w(P)$ tend to infinity or zero. Therefore $\Omega(Q)$ and $\Omega(P)$ tend to $+1$ or -1. For example, if $Q, P \in P_+$ and $\lambda(Q) = i\tau,\ \ \lambda(P) = i\theta$ where $\tau, \theta \to +\infty$, then

$$\Omega(Q) \to -1, \qquad\qquad \Omega(P) \to -1;$$

and formula 3.1 becomes

$$\{\mathcal{X}(Q), \mathcal{X}(P)\} \sim +2 \times \frac{(\mathcal{X}(Q) - \mathcal{X}(P))^2}{\lambda - \mu}.$$

Thus the formula asymptotically coincides with the bracket for the classical Weyl function in the open upper/lower hulf-plane, see [22].

Remark 3.7. The deformed AH bracket is invariant under linear–fractional transformations. Namely, if we introduce the new Weyl type function $\tilde{\mathcal{X}}$ by formula 2.26, then the bracket for $\tilde{\mathcal{X}}$ is given by the same formula 3.1. The proof is identical to the pure AH case, see [22]. Another consequence of invariance is the formula for the bracket of the first component $A(x, Q)$. Note, first

$$A(x, Q) = \frac{1}{1 + \mathcal{X}(x, Q)}.$$

This implies that $A(x, Q)$ is the meromorphic function with $g + 1$ zeros at

$$\boldsymbol{\sigma}_1(x), \ldots, \boldsymbol{\sigma}_g(x), \boldsymbol{\sigma}_{g+1}.$$

Therefore, for Q, P different from poles of A and also branch and singular points we have

$$\{A(x, Q), A(x, P)\} = -2 \times \frac{(A(x, Q) - A(x, P))^2}{\lambda(Q) - \lambda(P)} \times \frac{\Omega(Q) + \Omega(P)}{2}. \tag{3.25}$$

3.2. **Computation of the Poisson bracket for field variables.** The goal of this section is to show that the deformed AH bracket 3.1 can be taken as a starting point for the construction of the Poisson formalism. The inverse spectral transform 2.15 maps the deformed AH bracket to the phase space. Therefore, the bracket for the field variables $\psi(x)$ and $\overline{\psi}(x)$ can be obtained from the formula for the deformed AH bracket. Namely, we prove the following theorem

Theorem 3.8. *The deformed AH bracket for the Weyl function implies the follwowing Poisson brackets for the field variables*[11]*:*

$$\{\psi(z), \psi(y)\} = 0, \tag{3.26}$$
$$\{\overline{\psi}(z), \overline{\psi}(y)\} = 0, \tag{3.27}$$
$$\{\overline{\psi}(z), \psi(y)\} = 2i\delta(z - y). \tag{3.28}$$

[11]The identities are understood in the sense of generalized functions: $u(x) = v(x)$ if for any $f(x) \in C_0^\infty$ we have $\int u(x) f(x) dx = \int v(x) f(x) dx$.

These identities are an equivalent form of the Poisson bracket 2.2.

We use Remarks 2.3, 3.6 and the strategy developed in [22] for the proof of the analogous result.

Unfortunately, the inverse spectral transform 2.15 is very implicit. However, if the function $\mathcal{X}(x, Q)$ is known for all x then 2.24-2.25 imply the formulas for the potential:

$$\lim_{Q \to P_+} \frac{\lambda(Q)}{\mathcal{X}(x, Q)} = -i\overline{\psi}(x) \tag{3.29}$$

and

$$\lim_{Q \to P_-} \lambda(Q)\mathcal{X}(x, Q) = i\psi(x). \tag{3.30}$$

The limits are complex conjugate of each other due to 2.23 and

$$\begin{aligned}\lim_{Q \to P_-} \lambda(Q)\mathcal{X}(x, Q) &= \lim_{Q \to P_-} \lambda(\epsilon_a \epsilon_a Q)\mathcal{X}(x, \epsilon_a \epsilon_a Q) = \lim_{Q \to P_-} \overline{\frac{\lambda(\epsilon_a Q)}{\mathcal{X}(x, \epsilon_a Q)}} \\ &= \lim_{Q \to P_+} \overline{\frac{\lambda(Q)}{\mathcal{X}(x, Q)}}.\end{aligned}$$

Formulas 3.29–3.30 allow us to effectively solve the inversion problem

$$(\Gamma, \mathcal{X}(x, Q),\ x \in \mathbb{R}^1) \longrightarrow \mathcal{M}. \tag{3.31}$$

This approach requires the formula for the bracket $\{\mathcal{X}(x, Q), \mathcal{X}(y, P)\}$, for $x \neq y$. This bracket is computed asymptotically when one of the points Q or P approaches infinity from imaginary direction, see Remark 2.3.

Lemma 3.9. *Let $Q \to P_\pm$ and the point P is fixed, then*

$$\{\mathcal{X}(y, Q), \mathcal{X}(x, P)\} \sim e^{-i\lambda(Q)(x-y)}\{\mathcal{X}(x, Q), \mathcal{X}(x, P)\}. \tag{3.32}$$

Proof. The proof is analogous to the corresponding result for the pure AH bracket, [22]. The identity

$$\boldsymbol{e}(x, y, Q) = M(x, y, \lambda)\boldsymbol{e}(y, y, Q), \qquad \lambda = \lambda(Q),$$

implies

$$\mathcal{X}(x, Q) = \frac{m_{22}(x, y, \lambda)\mathcal{X}(y, Q) + m_{21}(x, y, \lambda)}{m_{12}(x, y, \lambda)\mathcal{X}(y, Q) + m_{11}(x, y, \lambda)}.$$

Note, when $Q \to P_\pm$,

$$M(x, y, \lambda) \sim e^{-\frac{i\lambda}{2}\boldsymbol{\sigma}_3(x-y)}.$$

Therefore,

$$\mathcal{X}(x, Q) \sim e^{i\lambda(Q)(x-y)}\mathcal{X}(y, Q).$$

The asymptotic in not uniform in the x-variable. Thus the left hand side is periodic in x but the expression at the right is not! □

The next lemma establishes that the Poisson tensor is real.

Lemma 3.10. *The Poisson brackets for the field variables $\psi(x)$ and $\overline{\psi}(x)$ are real*

$$\begin{aligned}
\overline{\{\psi(y), \psi(z)\}} &= \{\overline{\psi}(y), \overline{\psi}(z)\}, \\
\overline{\{\overline{\psi}(y), \psi(z)\}} &= \{\psi(y), \overline{\psi}(z)\}.
\end{aligned}$$

Proof. The proof requires the previous Lemma. The rest is analogous to the corresponding result for the pure AH bracket, [22]. □

Now we are ready to prove the main result.

Proof of Theorem 3.8. Due to Lemma 3.10, identities 3.26 and 3.27 are equivalent. We compute the bracket 3.26. Let $Q, P \in \Gamma_R$. Using formula 3.30 and Lemma 3.9 for $f(x) \in C_0^\infty$ and $y \leq z$ we have:

$$\int_{-\infty}^{z} dy f(y)\{\psi(z), \psi(y)\} = -\int_{-\infty}^{z} dy f(y)\{i\psi(z), i\psi(y)\}$$

$$\begin{aligned}
&= -\lim_{Q,P\to P_-} \int_{-\infty}^{z} dy f(y)\{\lambda(Q)\mathcal{X}(z,Q), \lambda(P)\mathcal{X}(y,P)\} \\
&= -\lim_{Q,P\to P_-} \lambda(Q)\lambda(P) \int_{-\infty}^{z} dy f(y)\{\mathcal{X}(z,Q), \mathcal{X}(y,P)\} \\
&= -\lim_{Q,P\to P_-} \lambda(Q)\lambda(P)\{\mathcal{X}(z,Q), \mathcal{X}(z,P)\} \int_{-\infty}^{z} dy f(y) e^{-i\lambda(P)(z-y)}.
\end{aligned}$$

Let Q, P be such that $\lambda(Q) = -i\tau,\ \lambda(P) = -2i\tau$. Since $\mathcal{X}$ has a zero at P_- we have using Theorem 3.1 when $\tau \to +\infty$,

$$\lim_{Q,P\to P_-} \lambda(Q)\lambda(P)\{\mathcal{X}(z,Q), \mathcal{X}(z,P)\} \sim \lambda(Q)\lambda(P)\frac{(\mathcal{X}(z,Q) - \mathcal{X}(z,P))^2}{\lambda(Q) - \lambda(P)} = O(\tau^{-1}).$$

For the integral we have

$$\int_{-\infty}^{z} dy f(y) e^{-i\lambda(P)(z-y)} = O(\tau^{-1}).$$

Therefore,

$$\{\psi(z), \psi(y)\} = 0, \qquad\qquad y \leq z.$$

Using skew symmetry of the bracket and interchanging y and z we obtain

$$\{\psi(z), \psi(y)\} = 0, \qquad\qquad y \geq z.$$

Taking the sum of these two formulas, we obtain 3.26.

Now we compute the bracket 3.28. Let $Q \to P_+,\ P \to P_-$. Then using formulas 3.29, 3.30 and Lemma 3.9 for $f(x) \in C_0^\infty$ and $y \le z$ we have:

$$\int_{-\infty}^{z} dy f(y)\{\overline{\psi}(z), \psi(y)\} = \int_{-\infty}^{z} dy f(y)\{-i\overline{\psi}(z), i\psi(y)\}$$

$$= \lim \int_{-\infty}^{z} dy f(y)\{\frac{\lambda(Q)}{\mathcal{X}(z,Q)}, \lambda(P)\mathcal{X}(y,P)\}$$

$$= \lim -\frac{\lambda(Q)\lambda(P)}{\mathcal{X}^2(z,Q)} \int_{-\infty}^{z} dy f(y)\{\mathcal{X}(z,Q), \mathcal{X}(y,P)\}$$

$$= \lim -\frac{\lambda(Q)\lambda(P)}{\mathcal{X}^2(z,Q)}\{\mathcal{X}(z,Q), \mathcal{X}(z,P)\} \int_{-\infty}^{z} dy f(y) e^{-i\lambda(P)(z-y)}.$$

Using Theorem 3.1,

$$\{\mathcal{X}(z,Q), \mathcal{X}(z,P)\} = -2 \times \frac{(\mathcal{X}(Q) - \mathcal{X}(P))^2}{\lambda(Q) - \lambda(P)} \times \frac{\Omega(Q) + \Omega(P)}{2}.$$

Since $\mathcal{X}$ has a pole at P_+ and a zero at P_-, we have asymptotically

$$-\frac{\lambda(Q)\lambda(P)}{\mathcal{X}^2(z,Q)} \quad \{\mathcal{X}(z,Q), \mathcal{X}(z,P)\} \sim$$

$$\sim -\frac{\lambda(Q)\lambda(P)}{\mathcal{X}^2(z,Q)} \times -2 \times \frac{\mathcal{X}^2(z,Q)}{\lambda(Q) - \lambda(P)} \times \frac{\Omega(Q) + \Omega(P)}{2}.$$

Let $P = \epsilon_a Q$ and $\lambda(Q) = i\tau,\ \tau \to +\infty$; then

$$\Omega(Q) \to -1, \qquad \Omega(P) \to -1;$$

and

$$-\frac{\lambda(Q)\lambda(P)}{\mathcal{X}^2(z,Q)} \quad \{\mathcal{X}(z,Q), \mathcal{X}(z,P)\} \sim -\frac{2\lambda(Q)\lambda(P)}{\lambda(Q) - \lambda(P)}.$$

Using steepest decent we have

$$\therefore \ = \lim_{\tau \to +\infty} -\frac{2\tau^2}{i\tau + i\tau} \int_{-\infty}^{z} dy f(y) e^{-\tau(z-y)} = i f(z).$$

Therefore,

$$\{\overline{\psi}(z), \psi(y)\} = i\delta(z-y), \qquad y \le z. \tag{3.33}$$

Using the realness of the bracket, by Lemma 3.10 we have

$$\{\psi(z), \overline{\psi}(y)\} = -i\delta(z-y), \qquad y \le z.$$

By the skew symmetry of the bracket, interchanging z and y,

$$\{\overline{\psi}(z), \psi(y)\} = i\delta(z - y), \qquad z \leq y. \tag{3.34}$$

Taking the sum of 3.33 and 3.34, we obtain 3.28.

□

3.3. **Functions of the first component.** The moving poles of the function $\mathcal{X}$ do not seem to be an appropriate object for the construction of canonical coordinates. Indeed there are g complex poles but $g+1$ real degrees of freedom in the isospectral set of potentials. The trouble is that the function $\mathcal{X}$ does not carry explicitly the information about the real poles of the Baker–Akhiezer functions. Both components have the same poles and this information disappears when we divide one component of the Baker–Akhiezer functions by another. What will happen if one will take another meromorphic function of the first component? Some natural choices are considered in this section.

The Floquet solution $\boldsymbol{e}(x, y, Q)$ satisfies the identity

$$[J\partial_x - JV]\, \boldsymbol{e}(x, y, Q) = 0, \qquad J = i\boldsymbol{\sigma}_2;$$

which is just another way to write an auxiliary spectral problem 2.4. Let us define the dual Floquet solution $\boldsymbol{e}^+(x, y, Q) = \left[e_1^+(x, y, Q), e_2^+(x, y, Q)\right]$ at the point Q by

$$\boldsymbol{e}^+(x, y, Q) = \boldsymbol{e}(x, y, \epsilon_\pm Q)^T.$$

The dual Floquet solution $\boldsymbol{e}^+(x, y, Q)$ satisfies[12]

$$\boldsymbol{e}^+(x, y, Q)\, [J\partial_x - JV] = 0.$$

Introduce the Wronskian function $\mathcal{W}(y, Q)$ by the formula

$$\mathcal{W}(y, Q) = \boldsymbol{e}^+(x, y, Q) J \boldsymbol{e}(x, y, Q).$$

The fact that $\mathcal{W}$ does not depend on x can be verified by differentiation. Evidently,

$$\mathcal{W}(y, Q) =< \boldsymbol{e}^+(x, y, Q) J \boldsymbol{e}(x, y, Q) >= \frac{1}{2l}\int_{-l}^{l} \boldsymbol{e}^+(x, y, Q) J \boldsymbol{e}(x, y, Q)\, dx.$$

As it follows from 2.20–2.21,

$$\mathcal{W}(y, Q) = A(y, \epsilon_\pm Q) - A(y, Q). \tag{3.35}$$

Now we list the properties of $\mathcal{W}(y, Q)$ which follow from Lemma 2.6 for finite gap potentials.

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[12]The action of the differential operator $D = \sum_{j=0}^{k} \omega_j \partial^j$ on the row vector f^+ is defined as $f^+ D = \sum_{j=0}^{k} (-\partial)^j (f^+ \omega_j)$.

- The function $\mathcal{W}(y,Q)$ is meromorphic on Γ and satisfies the identity

$$\begin{aligned}\mathcal{W}(y,\epsilon_\pm Q) &= -\mathcal{W}(y,Q),\\ \mathcal{W}(y,\epsilon_a Q) &= -\overline{\mathcal{W}(y,Q)}.\end{aligned}$$

- The function $\mathcal{W}(y,Q)$ has $2(g+1)$ poles, on both sheets at $\boldsymbol{\gamma}_k(y)$ and $\epsilon_\pm\boldsymbol{\gamma}_k(y)$ on the real ovals, and $2(g+1)$ zeros at the branch points $\boldsymbol{\lambda}_k^\pm$.
- At infinities $\mathcal{W}(y,Q)$ has the asymptotic behavior

$$\begin{aligned}\mathcal{W}(y,Q) &= +1+\frac{i\overline{\psi}(y)-i\psi(y)}{\lambda}+O\left(\frac{1}{\lambda^2}\right), && Q\in(P_+),\\ \mathcal{W}(y,Q) &= -1-\frac{i\overline{\psi}(y)-i\psi(y)}{\lambda}+O\left(\frac{1}{\lambda^2}\right), && Q\in(P_-).\end{aligned}$$

Let us introduce another function

$$\Xi(y,Q)=A(y,\epsilon_\pm Q)+A(y,Q).$$

In the next theorem the bracket for $\mathcal{W}(x,Q)$ will be expressed with the help of the function $\Xi(y,Q)$ and the traditional $\Omega(Q)$ and $\lambda(Q)$. This result will be used for construction of canonical variables.

Theorem 3.11. *Suppose $Q=(\lambda,w(Q))$ and $P=(\mu,w(P))$ are not the branch or crossing points of Γ; they are not poles of the functions $\mathcal{W}(Q)=\mathcal{W}(x,Q)$ and $\Xi(P)=\Xi(x,P)$. Then the bracket 2.2 for the functions $\mathcal{W}(Q)$ and $\Xi(P)$ is given by the formulas*

$$\{\mathcal{W}(Q),\mathcal{W}(P)\}=-2\times\frac{(\Xi(Q)-\Xi(P))(\mathcal{W}(P)\Omega(Q)-\mathcal{W}(Q)\Omega(P))}{\lambda-\mu},$$

$$\{\Xi(Q),\Xi(P)\}=-2\times\frac{(\Xi(Q)-\Xi(P))(\mathcal{W}(P)\Omega(P)-\mathcal{W}(Q)\Omega(Q))}{\lambda-\mu}.$$

Proof. We compute the bracket directly using formula 3.25. □

3.4. **Canonical variables.** The present section relates our results with approach of Novikov, Veselov and Dubrovin, [20, 6]. The next result is well known.

Theorem 3.12. [3]. *If $\boldsymbol{\gamma}_k=\boldsymbol{\gamma}_k(y)$, $k=1,\ldots,g+1$, and $y\in\mathbb{R}^1$, then the following identities hold:*

$$\{\lambda(\boldsymbol{\gamma}_k),\lambda(\boldsymbol{\gamma}_n)\} = 0, \tag{3.36}$$

$$\{p(\boldsymbol{\gamma}_k),p(\boldsymbol{\gamma}_n)\} = 0, \tag{3.37}$$

$$\{p(\boldsymbol{\gamma}_k),\lambda(\boldsymbol{\gamma}_n)\} = \delta_n^k. \tag{3.38}$$

Proof of Theorem 3.12 (incomplete). When Q is not a branch or crossing point, then $\lambda(Q)$ plays the role of a local parameter. If $\boldsymbol{\gamma}_k$, $\boldsymbol{\gamma}_n$ are such points, then the function $\mathcal{W}(x, Q)$ in the vicinity of these points can be written as

$$\mathcal{W}(x, Q) = \frac{\varphi_k(x, \lambda(Q))}{\lambda(Q) - \lambda(\boldsymbol{\gamma}_k)}, \qquad Q \in (\boldsymbol{\gamma}_k);$$

and

$$\mathcal{W}(x, P) = \frac{\varphi_n(x, \lambda(P))}{\lambda(P) - \lambda(\boldsymbol{\gamma}_n)}, \qquad P \in (\boldsymbol{\gamma}_n).$$

Then, by standard properties of the Poisson bracket,

$$\{\mathcal{W}(x, Q), \mathcal{W}(x, P)\} = \frac{\{\lambda(\boldsymbol{\gamma}_k), \lambda(\boldsymbol{\gamma}_n)\}}{(\lambda(Q) - \lambda(\boldsymbol{\gamma}_k))^2(\lambda(P) - \lambda(\boldsymbol{\gamma}_n))^2} + \dots.$$

Dots signify terms of order lower than four when $Q \to \boldsymbol{\gamma}_k$ and $P \to \boldsymbol{\gamma}_n$. From another side, using the result of Theorem 3.11 we have

$$\{\mathcal{W}(x, Q), \mathcal{W}(x, P)\} = \frac{O(1)}{\lambda(Q) - \lambda(\boldsymbol{\gamma}_k)} + \frac{O(1)}{\lambda(P) - \lambda(\boldsymbol{\gamma}_n)}.$$

The highest order of the pole on the right is one. These imply the first identity 3.36. $\square$

Evidently we did not use finite-gap property anywhere in the proof. It seems to be an important task to complete the proof of the Theorem using our approach. We will return to this question elsewhere.

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